Black Hole thermodynamics

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Contents

1 Thermodynamic Preliminaries

Here we recap the thermodynamic postulates from Kammerlander and Renner [1][2].

Postulate 1 (Thermodynamic systems)**.** *We denote the* thermodynamic world *by* Ω*. The set of thermodynamic systems is given by the finite and non-empty subsets of the thermodynamic world* $\mathcal{S} = \{S \subset \Omega |\ 0 < |S| < \infty\}$*.*

Postulate 2 (Thermodynamic processes)**.** *The set of* thermodynamic processes *is denoted by* $\mathcal{P}$ *and is non empty. A thermodynamic process* $p \in \mathcal{P}$ *specifies the* initial and final state *of the finite and nonzero involved atomic systems* $A \in \mathcal{A}$ *by means of the functions* $\lfloor \cdot \rfloor_A : \mathcal{P} \to \Sigma_A$ *and* $\lceil \cdot \rceil_A : \mathcal{P} \to \Sigma_A$*. For systems not involved in a process the functions* $\lfloor \cdot \rfloor_A$ *and* $\lceil \cdot \rceil_A$ *are not defined. We denote the set of atomic systems involved in* p *by* $\mathcal{A}_p$*.*

Postulate 3 (Concatenated processes)**.** *Let* p, p' *be two processes such that for all* $A \in \mathcal{A}_p \cap \mathcal{A}_{p'}$ *it holds that* $\lceil p \rceil_A = \lfloor p' \rfloor_A$*. These processes can be* concatenated *to a new process* $p' \circ p \in \mathcal{P}$*. In case that* $\mathcal{A}_p \cap \mathcal{A}_{p'} = \emptyset$*, the concatenation commutes, i.e.* $p' \circ p = p \circ p'$*. An atomic system* $A \in \mathcal{A}$ *is involved* $p' \circ p$ *if* $A \in \mathcal{A}_p \cup \mathcal{A}_{p'}$*. For these systems the initial state is given by*

$$\lfloor p' \circ p \rfloor_A = \begin{cases} \lfloor p \rfloor_A & \text{if } A \in \mathcal{A}_p \\ \lfloor p' \rfloor_A & \text{otherwise} \end{cases} \tag{1.1}$$

and the final state is given by

$$\lceil p' \circ p \rceil = \begin{cases} \lceil p' \rceil & \text{if } A \in \mathcal{A}'_p \\ \lceil p \rceil & \text{otherwise} \end{cases} . \tag{1.2}$$

Postulate 4 (Work)**.** *For any atomic system* $A \in \mathcal{A}$*, there exists a* work function $W_A : \mathcal{P} \to \mathbb{R}$*. This function maps a process* $p \in \mathcal{P}$ *to the work* $W_A(p)$ *done on the atomic system* A*. We use the sign convention that* $W_A(p)$ *is positive, whenever work is done on* A *while executing* p*. If* A *is not involved in* p *then* $W_A(p) = 0$*.*

Postulate 5 (Work function under concatenation)**.** *Let* $p, p' \in \mathcal{P}$ *be two processes such that* $p \circ p'$ *is well-defined, then the work cost of the concatenated system for an atomic system* $A \in \mathcal{A}$

$$W_A(p \circ p') = W_A(p) + W_A(p'). \tag{1.3}$$

Postulate 6 (Freedom of description)**.** *Let* $S, C \in \mathcal{S}$ *be two disjoint systems, and let* $p \in \mathcal{P}_{S \vee C}$ *be a process that is catalytic on* C*. Then, there exists a process* $\tilde{p} \in \mathcal{P}_S$ *such that the initial and final states of* S *are the same for both processes and the work done on system* S *is the same for both processes and all quantities* B*:*

$$\begin{aligned} \lfloor \tilde{p} \rfloor_S &= \lfloor p \rfloor_S \\ \lceil \tilde{p} \rceil_S &= \lceil p \rceil_S \\ W_S(\tilde{p}) &= W_S(p). \end{aligned} \tag{1.4}$$

Postulate 7 (First law)**.** *For any system* S *the following statements hold:*

1. *For any pair of states $\sigma_1, \sigma_2 \in \Sigma_S$ there exists either a work process $p \in \mathcal{P}_S$ such that $\lfloor p \rfloor_S = \sigma_1$ and $\lceil p \rceil_S = \sigma_2$ or a work process $p' \in \mathcal{P}_S$ such that $\lfloor p' \rfloor_S = \sigma_2$ and $\lceil p' \rceil_S = \sigma_1$.*
2. *The total work cost of any work process $p \in \mathcal{P}$ $W_S(p)$ only depends on the initial and final state of the process and not on other properties of the process.*

Postulate 8 (Equivalent systems)**.** *Let $A \in \mathcal{A}$ be an arbitrary atomic system and let $n \in \mathbb{N}$ be a natural number. Then there exists n different equivalent atomic systems $A_1 \hat{=} A_2 \hat{=} ... \hat{=} A_n \hat{=} A$.*

Postulate 9 (Second law for a single quantity)**.** *Consider a quantity A heat reservoir $R \in \mathcal{R}$ together with an arbitrary system $S \in \mathcal{S}$ and a work process $p \in \mathcal{P}_{R \vee S}$ which acts on the composite system. If p is cyclic on S, then*

$$W_S(p) \geq 0. \tag{1.5}$$

Postulate 10 (Carnot processes)**.** *Let R_1 and R_2 be reservoirs. For all heats $Q \in \mathbb{R}$ there exists a system S and a reversible work process $p \in \mathcal{P}_{S \vee R_1 \vee R_2}$ such that p is cyclic on S and $Q_{R_1}(p) = Q_A$ holds.*

Postulate 11 (Reversible processes)**.** *Let S be a system and σ_1, σ_2 two states of the system S. Then there exists a finite sequence of reversible processes $\{p_i\}_i \in \mathcal{P}_{S \vee R_i}$ such the concatenation of the processes $p = p_1 \circ ... \circ p_N$ is defined. Additionally, each process acts on the system, only on one reservoir R_i.*

2 Black holes

Before we discuss the thermodynamics of black holes, we first give an short review of black holes where we will first discuss the Schwarzschild metric, then the Kerr-Newman metric, and finally we will consider Penrose diagrams.

2.1 Schwarzschild metric

The Schwarzschild solution describes the situation outside of a spherically symmetric mass, which a non-rotating, neutral black hole is. Furthermore, the Schwarzschild metric is the most general spherically symmetric, static, and asymptotically flat solution to the Einstein equation. The metric is given by

$$\begin{aligned} ds^2 &= -\frac{r-2GM}{r}dt^2 + \frac{r}{r-2GM}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \\ &= -\frac{r-2GM}{r}dt^2 + \frac{r}{r-2GM}dr^2 + r^2 d\Omega_2^2 \end{aligned} \tag{2.1}$$

where $d\Omega_2$ is the unit metric on the two sphere $\mathbb{S}_2$. For a derivation of the Schwarzschild metric see [3, Chapter 6.1]. The curvature of the Schwarzschild metric can be visualized with Flamm's paraboloid [4] as depicted in Figure 1. In the above equation, G is the gravitational constant and M is a parameter with units of mass. Note, that M is at this point simply a parameter of the solution and does not need to be the mass of the spherically symmetric body. However, later we will see that one can indeed identify the parameter M with the black hole mass and there is a corresponding energy conservation law. A first indicator for this is that for $r \gg 2GM$ the geodesic equation for a massive particle of mass m reduces to the standard Newtonian equation of motion of a particle and a spherically symmetric mass with mass M [5]:

$$m\ddot{\vec{x}} = -\frac{GmM}{r^2}\vec{e_r}. \tag{2.2}$$

2.1.1 Singularities

Let us discuss certain properties of the Schwarzschild metric. This metric has two points where the coefficients diverge, namely at $r = 0$ and $r = 2GM$. Singularities in the metric components can have two origins. On one hand, they could stem from the space time being indeed singular. On the other hand, space time could also be non-singular, but the chosen coordinates cannot properly describe that region of space time. In the case of the Schwarzschild metric, we encounter both cases. The singularity at $r = 0$ is a real space time singularity, as the fully contracted Riemann tensor $R_{abcd}R^{abcd}$ diverges at $r \to 0$. This tensor is a measure for the strength of tidal forces. This means that any object in the vicinity of $r = 0$ would be pulled apart by the strong tidal forces. On the other hand, the singularity at $r = 2GM$ is due to a 'bad' choice of coordinates. There are no quantities which diverge at $r = 2GM$[1] [3, 5].

Even though there is no space time singularity at $r = 2GM$, there are interesting properties of the metric in that region. At $r = 2GM$ the coefficients corresponding to dr^2 and dt^2 change signs. This means that r becomes timelike and t becomes spacelike. Therefore, for $r < 2GM$

[1]For a choice of coordinates where there is no singularity at $r = 2GM$ see [5] section 2.2.

one cannot prevent falling further into the black hole, just as one cannot prevent traveling further in time. Therefore, nothing can come out of the surface $r = 2GM = r_s$, not even massless particles such as light.

Definition 2.1 (Event horizon)**.** *We call a surface an event horizon if it is the boundary of a space time region where every point in that region cannot send a signal to a timelike geodesic.*

For the Schwarzschild geometry of space time this occurs for geodesics with $r > 2GM$ and points with $r < 2GM$. Therefore, we call the surface $r = 2GM$ the event horizon of a black hole and $r_s = 2GM$ the Schwarzschild radius. Moreover, if a spherically symmetric mass is compacted to below its Schwarzschild radius it will also become a black hole [5].

2.1.2 Red shift

Another important feature of the Schwarzschild metric is the gravitational red shift. Let us consider a curve with fixed r. Then, a fixed interval of coordinate time t will result in less proper time s for r getting closer to $2GM$, because the pre-factor

$$\frac{r - 2GM}{r} \tag{2.3}$$

of dt^2 gets smaller. Therefore, an observer at infinity will see the clocks of an in-falling observer slow down and the frequency of a light signal sent out will be red shifted. More precisely a signal of energy E for a in-falling observer will be red shifted to

$$\sqrt{\frac{r - 2GM}{r}} E = \alpha E \tag{2.4}$$

for an observer at infinity [5].

2.2 Kerr-Newman metric

So far the black holes that are described by the Schwarzschild solution cannot rotate and have no charge. The solutions of charged and rotating black holes are given by the Kerr-Newman metric. The rotating solutions where found by Kerr [6] and later generalized by Newman et al. [7] to charged solutions. These solutions are parameterized by three parameters (M, a, e) and given by

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2(\theta)}{\Sigma}\right) dt^2 - \frac{2a\sin^2(\theta)\left(r^2 + a^2 - \Delta\right)}{\Sigma} dt d\phi \\ &+ \left(\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2(\theta)}{\Sigma}\right) \sin^2(\theta) d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \Sigma &= r^2 + a^2\cos^2(\theta) \\ \Delta &= r^2 + a^2 + e^2 - 2Mr. \end{aligned} \tag{2.6}$$

Let us consider some properties of this solutions. In case $a = e = 0$, the Kerr-Newman metric reduces to the Schwarzschild metric. Additionally, it is stationary, axis symmetric and asymptotically flat, that means for $r \to \infty$ the space time geometry approaches Minkowski space in spherical coordinates [3].

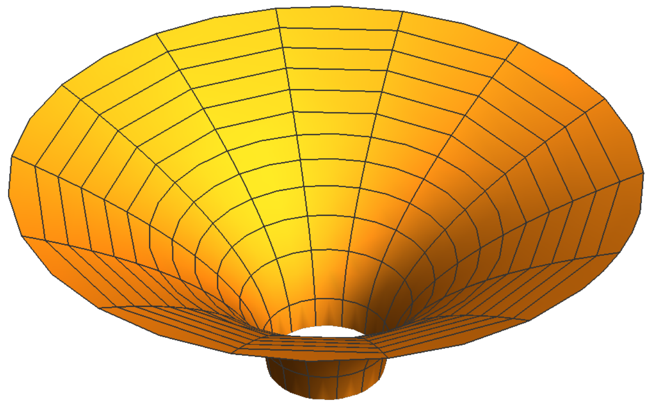

Figure 1: Visualization of the curvature of the Schwarzschild metric outside of the event horizon, called Flamm's paraboloid [4]. This visualization considers Schwarzschild space for $t = const., \theta = \pi/2$, the surface as a function of the remaining coordinates r, ϕ is given by $w = 2\sqrt{r_s(r - r_s)}$. This surface has the property that distances measured on the paraboloid correspond to distances measured in the Schwarzschild metric.

2.2.1 Singularities

Kerr-Newman metric has a diverging components at $\Sigma = 0$ and at $\Delta = 0$. The latter, in turn, is an artifact of the choice of coordinates, whereas the former is a real singularity of space time. There is one peculiarity about this singularity at $\Sigma = r^2 + a^2 \cos^2(\theta) = 0$. It seems like this is only a singularity if $\theta = \frac{\pi}{2}$, as only there the cosine term in Σ is zero. In particular, this would mean that if we approach $r = 0$ along a geodesic with $\theta = 0$ the curvature would remain finite. However, this is merely an artifact of the choice of coordinates and it can be shown that for every geodesic ending up in $r = 0$ the curvature blows up. A summary of how to see this can be found in [3, p. 315] with more details in [8].

2.2.2 Ergosphere

The Kerr-Newman space time gives rise to the *ergosphere*, which is the region where the time translation Killing field $(\frac{\delta}{\delta t})^a$ becomes spacelike. This happens in the region

$$r_+ < r < M + (M^2 - e^2 - a^2 \cos^2(\theta)))^{1/2}. \tag{2.7}$$

The 'side view' of the ergosphere of a black hole can be seen in Figure 2 This region has several properties. First, in the ergosphere an observer cannot be stationary, even though this region is still outside of the event horizon [3]. Second, the energy of a particle at infinity is defined as the scalar product between $(\frac{\delta}{\delta t})^a$ and the four-momentum of the particle p^a. As $(\frac{\delta}{\delta t})^a$ is spacelike in the ergosphere, the energy can become negative. This property will be important once we consider processes acting on the black hole, as this will allow us to extract rotational energy from a black hole [9].

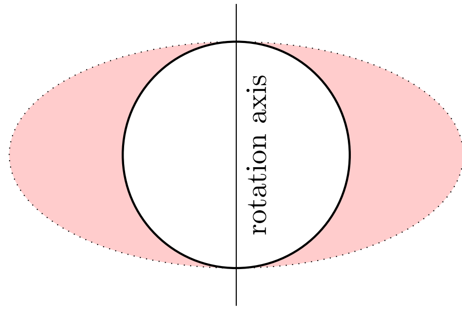

Figure 2: The 'side view' of the ergosphere (red) of a black hole.

2.3 Penrose diagrams

When studying space time geometries we are sometimes not interested in the details of space time, but only in its causal structure, which can be graphically represented with a so-called *Penrose diagram*. To compact often infinite space times in a finite diagram one uses the following fact:

> If for two metrics $g'_{\mu,\nu}(x), g_{\mu,\nu}(x)$ it holds that $g'_{\mu,\nu}(x) = e^{2\omega(x)}g_{\mu,\nu}(x)$ and ω is a smooth real function, then they have the same null-geodesics, i.e. light rays travels on the same path. Additionally, spacelike and timelike geodesics are mapped to space and timelike geodesics respectively.

Let us first consider Minkowski space time. In spherical coordinates the Minkowski metric is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2. \tag{2.8}$$

To represent the entire space time in a finite amount of space we need to compact the coordinates. Therefore, we consider the following coordinate transformation

$$T + R = \arctan(t + r) \tag{2.9}$$

$$T - R = \arctan(t - r). \tag{2.10}$$

In these coordinates the Minkowski metric is given by

$$ds^2 = \frac{1}{\cos^2(T+R)\cos^2(T-R)}(-dT^2 + dR^2 + \left(\frac{\sin(2R)}{2}\right)^2 d\Omega_s^2). \tag{2.11}$$

The ranges of these coordinates are given by $|T \pm R| < \frac{\pi}{2}, R \geq 0$, which is indeed finite. When we suppress a 2-sphere at each point, which does not result in a loss of much information as the Minkowski space is spherically symmetric, we can depict Minkowski space time in its Penrose diagram as shown on Figure 3. Because of the above fact, light rays still travel on lines with an angle of 45°. We label the following regions on the boundary: First, $i^\pm$ are the *past* and *future timelike infinities*, from which timelike geodesics 'start' and 'end' respectively. Second, $J^\pm$ are the *past* and *future null infinities*, from which null geodesics 'start' and 'end' respectively. Third, i^0 is the *spatial infinity* where spacelike geodesics 'end'.

The goal of introducing Penrose diagrams is to represent the causal structure of space time. Indeed, from this diagram we can immediately see that Minkowski space time has no singularities.

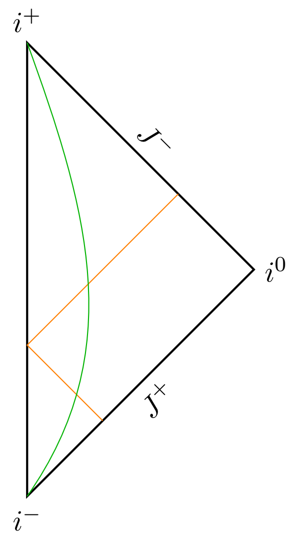

Figure 3: **The Penrose diagram of Minkowski space.** The orange line is the world line of a light ray. The green line is a timelike geodesic. $i^{\pm}$ are the past and future timelike infinities, where timelike geodesics 'start' and 'end'. $J^{\pm}$ are the past and future null infinities where the null geodesics 'start' and 'end'. i^0 is the spatial infinity where spacelike geodesics 'end'.

2.3.1 Black holes

The Penrose diagram of a Schwarzschild black hole is given by Figure 4. As light rays travel again on curves with an angle of 45°, we immediately see that from the black hole (upper triangle) no light can escape anymore, as for each point the entire future light cone lies within the triangle. The Schwarzschild metric is time reversal invariant, therefore there has to be the time reversed object of a black hole, the white hole, where things can only exit but not enter and where for each point the entire past light cone is located in the white hole, which means that nothing can enter the white hole.

2.3.2 Real black holes

The metrics we have considered so far have not taken into consideration how a real black hole forms. Real black holes are formed from a collapse of matter. Only after all of the effects of the collapse have settled down, will the black hole be in a stationary geometry.

Let us consider the non-rotating, non-charged case. To not be limited by the details of stellar collapses, we consider a spherically collapsing photon shell. The space time geometry inside the photon shell is given by the Minkowski geometry and outside by the Schwarzschild geometry. Therefore, to obtain the total Penrose diagram we need to stitch together the Minkowski space Penrose diagram and the Schwarzschild space Penrose diagram as depicted in Figure 4. This results in the Penrose diagram depicted in Figure 5. From the Penrose diagram we see that the event horizon is both inside and outside of the shell of photons. This means that we could be inside the event horizon of a black hole that has not formed yet, and we have no way of knowing whether this is the case or not [5].

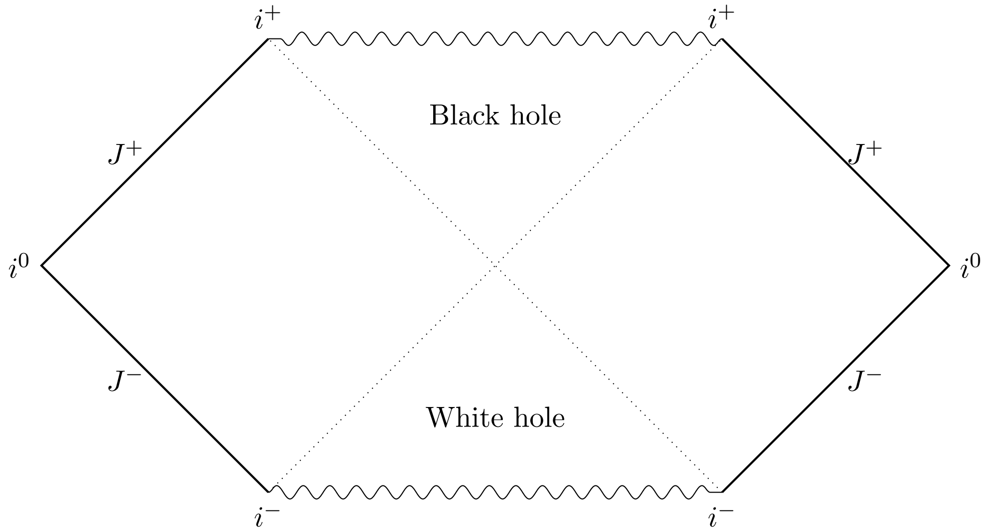

Figure 4: The Penrose diagram of a Schwarzschild black hole. The wavy lines are the singularities, where the suppressed two sphere $\mathbb{S}_2$ at each point shrinks to zero and the space time becomes singular. The dotted lines are the event horizons. The lower triangle is a white hole where things can only escape from, wheras the upper triangle is a black hole where nothing can escape from.

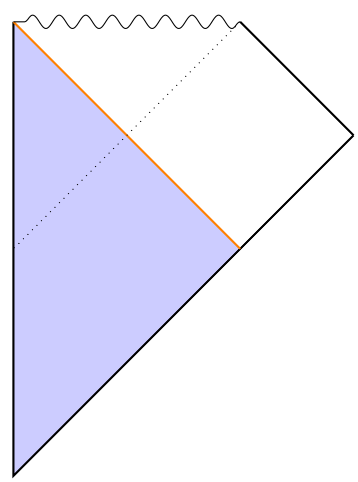

Figure 5: The Penrose diagram of a real black hole. The wavy line is the singularity, where the suppressed $\mathbb{S}_2$ shrinks to zero. The dotted line is the event horizon. The blue shaded region is Minkowski space and the orange line is the collapsing photon shell.

3 Black hole as a thermodynamic object

We take the same notion of how to see a black hole as a thermodynamic object as Wallace [10]. It is not obvious that black holes can be seen as a thermodynamic object. In particular, in literature (for example [8]) a black hole has been defined as a space time region from which no null geodesics can reach the future infinity. For such an object it does not make sense to consider the thermodynamics of it, as space time regions do not evolve in time. For example, it would then not make sense to define a process with initial and final states [10].

On the other hand, informally one often speaks about throwing something into a black hole and as a result the black hole is gaining mass, which implies an evolution of the black hole in time. Formally, this notion can be implemented with the membrane paradigm by [11, 12]. This paradigm considers a timelike surface, the *stretched horizon*, which is one Plank length larger than the event horizon. Practically, we can consider the stretched horizon as a one way surface, as only objects with a very large acceleration can still escape from it. In particular, light escaping from the stretched horizon would need to be described by Plank-scale physics. Moreover, the stretched horizon is just an ordinary timelike surface, whose evolution in time we can describe. Therefore, in thermodynamics of a black hole we consider the change of the stretched horizon as seen by an outside observer [10].

Surprisingly, the stretched horizon cannot just be used to make sense of a black hole as a thermodynamic object, but it has also local 'physical properties' even though there it is only a fictional notion. If one considers the stretched horizon as two dimensional electrically conducting viscous fluid where energy and charge are distributed in such a way that they cause the outside gravitational field, then one can use this model to make predictions for an outside observer. However, if one would actually fall to the stretched horizon one would not encounter anything as these things are purely fictional. For some examples see [10] and a more extensive discussion see [11].

3.1 State space

The no-hair conjecture [13] says that the unique stationary solutions in Einstein-Maxwell theory, which considers gravity and electrodynamics combined, are the Kerr-Newman solutions. Much work has been done to prove the conjecture, but there are still some questions open. However, according to Carter the conjecture has been proven to a 'degree of rigor usually considered acceptable in physics' [14].

Therefore, we only consider only these types of solutions, which are parameterized by three parameters $(M, L := aM, Q := e)$ with

$$Q^2 + L^2/M^2 \leq M^2. \tag{3.1}$$

In the following, we will only consider the case where there is actually a black hole, i.e. $M \neq 0$. Mathematically, the quantities M, L, Q are just parameters of the solution and do not need imply a conservation law. Fortunately, one can use the so called ADM formalism [15, 16] to define mass, angular momentum and charge of a black hole as a boundary integral over i^0 in the Penrose diagram. The resulting mass, angular momentum, and charge of the black hole are then, indeed, just M, L and Q. We only need one parameter for the angular momentum, as we assume that the angular momentum is aligned with the z-axis of the coordinate system. As space time becomes flat at $r \to \infty$ it is possible to define a global conservation law of these quantities and we can consider them conserved [17, 10].

4 Penrose processes

Now that we have established how we understand a black hole as a thermodynamic object, we can consider work processes that involve a black hole. Together, all of the processes we consider in this section fulfil the first point of the first law. The family of processes we consider are the Penrose processes. The first idea came from Penrose in [18] where he considers a process that can extract 'rotational energy' from a black hole. Penrose's original idea was then further developed by Christodoulou [19] and Penrose and Floyd [9]. First, we will discuss these processes for non-charged rotating black holes ($L \neq 0, Q = 0$) and then generalize the processes to charged rotating black holes.

4.1 Penrose process for rotating black holes

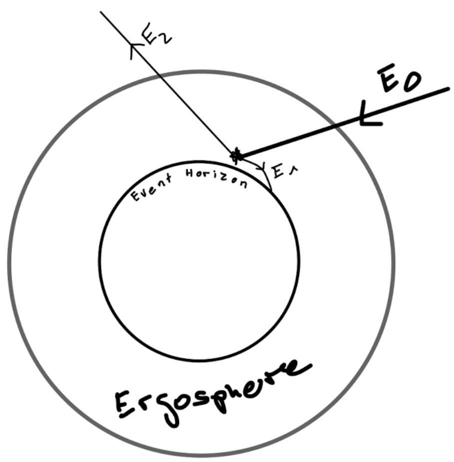

Figure 6: Conceptual depiction of the Penrose process, adapted from [19]. Contrary to the previous depiction of the ergosphere here we have a 'top view' of the black hole.

In the Penrose process, a particle with four-momentum p_0 is dropped from infinity into the black hole. In the ergosphere, this particle decays into two particles with four-momentum p_1, p_2. Then, we let particle 2 escape to infinity and particle 1 fall into the black hole. From conservation of angular momentum and energy we find that the new angular momentum L' and mass M' of the black hole are given by

$$L' = L_1 + L = L_0 - L_2 + L, \tag{4.1}$$

$$M' = E_1 + M = M_0 - M_2 + M \tag{4.2}$$

where M_i, L_i are the angular momentum and energy as seen from infinity of the particle with four-momentum p_i. As the decay happens in the ergosphere, the energy of particle 1 can be negative as seen from infinity. Furthermore, it is possible for the decay to be such that the particle 1 falls into the black hole and the particle 2 escapes to infinity. Graphically, such a process is depicted in Figure 6.

Let us consider the initial and final states of the black hole during this process. The black hole can be in any initial state

$$\lfloor p_{penrose}(L_1, E_1) \rfloor_{BH} = (M, L) \tag{4.3}$$

and the final state is then

$$\lceil p_{penrose}(L_1, E_1) \rceil_{BH} = (M + E_1, L + L_1), \tag{4.4}$$

if the decay in the process is described as above. During this process the work

$$W(p_{penrose}(L_1, E_1)) = E_0 - E_2 = E_1 \tag{4.5}$$

is done on the black hole, as this is the change of the energy of the particle that arrives at infinity again.

In general, Penrose processes are not reversible, but with the right parameters the process can also be done in a reversible manner. Let us first clarify what we mean by reversible. Naturally, the particle that has fallen in the black hole will never come out again. However, the process can be reversible in the sense of the thermodynamic framework, where we only require that there exists a second process which reverses the state of the black hole and does exactly the negative work on the black hole of the first process. In fact, a Penrose process is reversible if the decay happens at the event horizon and the radial velocity of the in-falling particle is zero. To see this we follow [19]. Let us consider the case where the angular momentum and energy of the in-falling particle can be seen as infinitesimal and let us consider a process where the mass of the black hole is reduced, i.e. $E_1 < 0$. Let us consider the energy E of a particle with a turning point at r angular momentum ϕ and mass μ

$$E^2(r^3 + (L/M)^2(r + 2M)) - 4MEL/M\phi + (2M - r)\phi^2 - \mu^2 r^2(r - 2M) - L/M\mu^2 r = 0. \tag{4.6}$$

The Penrose process is the most efficient if the reduction of the mass of the black hole is the largest for a given angular momentum of the particle, which means that E_1 needs to be as negative as possible. This happens at $r = r_+$, which is the event horizon of the black hole, then the energy is given by

$$E_1 = \frac{L/M}{r_+^2 + (L/M)^2} L_1. \tag{4.7}$$

In particular, for this to be negative the angular momentum of the in-falling particle needs to be in the opposite direction to the black hole. Repeating this process we can reach states which fulfill the relation

$$M^2 = M_{irr} + \frac{L^2}{4M_{irr}^2}. \tag{4.8}$$

This means that this process does not change the irreducible mass of the black hole. The irreducible mass of the black hole can be intuitively interpreted as the minimal mass a black hole has after extracting all rational energy of the black hole. To reverse the process, we apply the same process where the particle decays at the event horizon, but with the opposite energy and angular momentum, which also fulfils Equation (4.7). Together, these processes allow us to reversibly reach all states with the same irreducible mass as implicitly defined in the above equation, with parameters between $L = 0, M^2 = M_{irr}^2$ and $|L| = M^2, M^2 = 2M_{irr}^2$. Furthermore, these processes are quasistatic, as we can add as much or as little mass to the black hole as we want, with the differential work given by $\delta W = dE_1$ and the differential angular momentum change dL_1 are related to each other by

$$dE_1 = \frac{L/M}{r_+^2 + (L/M)^2} dL_1. \tag{4.9}$$

Another quasistatic Penrose-type process we can consider is one where the particle decays already at infinity and the in-falling particle does not carry any angular momentum – a quasistatic process on the curve $L = 0$. The differential work done during this process is given by $\delta W = dE_1$. The curves of this process and the reversible Penrose process are depicted in Figure 7.

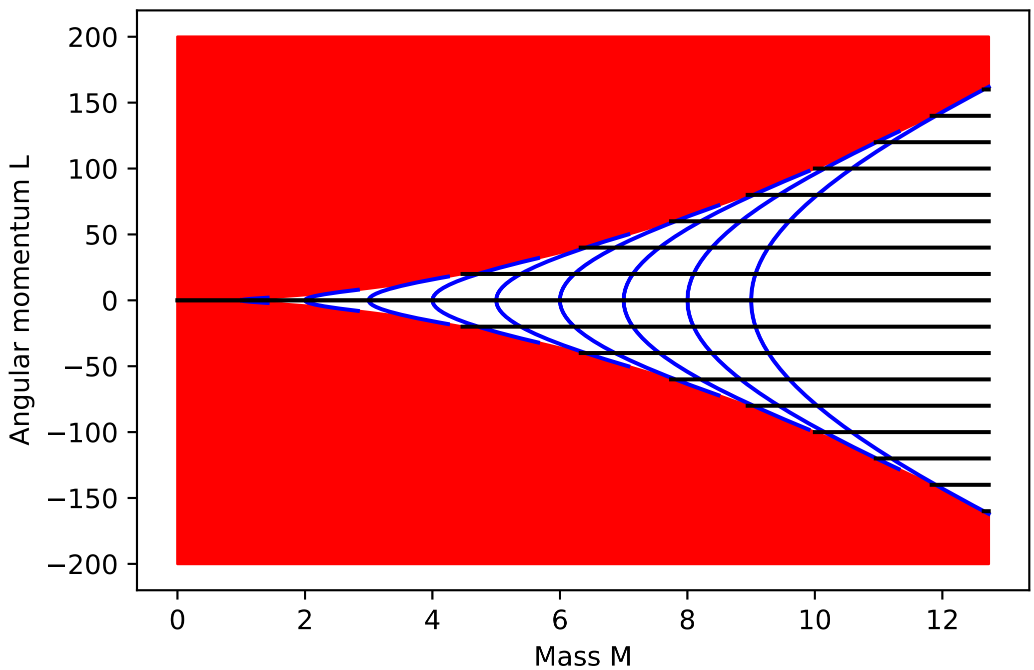

Figure 7: (L,M)-diagram of the work processes. The blue lines are the paths of the reversible Penrose processes for non-charged black holes and the black lines the paths of the irreversible variants used.

4.2 Penrose process for charged rotating black holes

A similar principle we used to introduce Penrose processes for rotating black holes can also be applied to charged black holes, originally proposed by [20]. The process is the same we have used for non-charged black holes, except that the decaying particle can also have a charge.

In the same way as for the rotating black hole, the process can be done reversibly when the particle decay happens at the event horizon with zero radial velocity such that the energy, charge, and angular momentum of the in-falling particle can be seen as infinitesimal. In this case the black hole evolves along a curve of constant irreducible mass, where for charged holes the irreducible mass is given implicitly by

$$M^2 = (M_{irr} + \frac{Q^2}{4M_{irr}})^2 + \frac{L^2}{4M_{irr}^2}. \tag{4.10}$$

Unlike the reversible Penrose process for neutral holes, when considering charged holes we have two parameters that can be chosen, as states of the same irreducible mass are a 2D surface in state space. Furthermore, the initial and final states of the process need to fulfil

$$\frac{L^2}{2M_{irr}^2} + \frac{Q^2}{16M_{irr}^2} \leq 1. \tag{4.11}$$

Same as the Penrose process for non-charged black holes, this process is quasistatic and the work done on the black hole is the same as before. The differential work done on the black hole is

$$\delta W = dE_1. \tag{4.12}$$

5 Internal energy

The processes we discussed in the previous section fulfil the first part of the first law. To see this, let us consider two states (M, L, Q) and (M', L', Q'). These two states can be connected with each other via Penrose processes. First we apply a reversible Penrose process such that $(M, L, Q) \to (M_{irr,1}, 0, 0)$ and then we apply the process where we just throw in mass into the black hole until the black hole reaches the irreducible mass of the target state. Then we again apply the reversible Penrose process to reach the state (M', L', Q'). Furthermore, the work done on the black hole in these two kinds processes is the mass difference between the start and end state, thus, the work only depends on the start and end state of the processes.

Note that the differential work for each of the above processes is expressed by the state change of the black hole state parameter M. Therefore, the value of each of the internal energy is given by just the state parameter M itself plus some arbitrary reference constants. This is not surprising, as the input theory to thermodynamics sees the black hole parameter M as the black hole's energy, and it is also conserved in the input theory. Therefore, for the work function to be compatible with the underlying theory, the work needs to compensate the change in the parameter M.

6 Hawking radiation

The second quantity we want calculate in the thermodynamic framework is the entropy. However, for this calculation we need a reversible processes with the potential involvement of a reservoir between all states, and if these processes would not exist, a black hole would violate Postulate 11, which requires there to be such processes connecting all two states. So far, we have seen that some Penrose processes are reversible. However, these processes cannot change the irreversible mass of a black hole. In fact, Hawking's area theorem [21], which says that for classical black holes the area of a black hole $A = 16\pi M_{irr}^2$ can only increase. Therefore, it is impossible for there to be a classical process decreasing the irreducible mass.

The black hole we have described so far does not emit any thermal radiation. Additionally, if we would place it in a box full of thermal radiation, it would eventually swallow all of the radiation. Therefore, the black hole seems to have a temperature of absolute zero. However, so far we have not taken into account quantum mechanics, where the Hawking effect will make the black hole radiate. This consideration will also make it possible for a black hole to coexist with radiation in a box.

6.1 Consequence of evaporation for black hole thermodynamics

Black hole radiation evaporates the black hole. However, the first law requires an identity process to exist for every black hole state. Therefore, Hawking radiation complicates the discussion of a black hole state, as doing nothing to the black hole no longer leaves the state invariant. The problem of radiation is not a problem only occurring with black holes, as any electromagnetically-interacting system placed in a vacuum also radiates and loses energy. To remedy this situation we can assume: either that all processes happen on a time scale much faster than the black hole evaporation and we neglect the change due to radiation; or we place the black hole in a box filled with radiation with the same temperature as the black hole [10].

7 Reversible process involving a black hole

In this section all black holes are non-charged black holes. To calculate the entropy of a black hole we need to find one more reversible process which can change the irreducible mass of a black hole. We use one of the processes used by Kaburaki and Okamoto [22] in their Carnot engine using the black hole as a working medium.

We cannot directly put a black hole in contact with a heat reservoir, as otherwise the black hole would evaporate away [23, 24, 25]. Therefore, we consider a black hole inside a finite rotating cylinder filled with radiation and fully reflective walls, like in Figure 8. The conditions for such a setup to not change over time are presented in [23]. In particular, the two systems are unchanging over time if

$$T_{rad} = T_{bh} \tag{7.1}$$

and for the angular velocity of the radiation Ω it holds that

$$\Omega = \Omega_{bh} := \frac{L}{2M^2(M + \sqrt{M^2 - L^2/M^2})}. \tag{7.2}$$

The process then can be divided into two steps. First, we change the angular momentum of the black hole by infinitesimally small amount with a reversible Penrose process while also

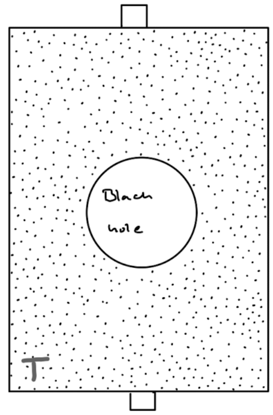

Figure 8: Setup of the isothermal process, adapted from [22].

changing the rotational velocity of the bath to match that of the black hole, and let the system evolve until it reaches a stationary state. In this step we do the work

$$\delta W_{bh} = \Omega dL, \tag{7.3}$$

on the black hole, which is the work done during the Penrose process, as we saw in Section 4.

In a next step, we connect the rotating gas to a reservoir of temperature $T_{res} = T_{bh}$ to let the gas have its original temperature again, but not long enough for this to affect the black hole. If these steps are made infinitesimally small, the total process is reversible.

The curve of this process in the black hole state space is given by an isotherm $T_{bh} = const$, as we use the external reservoir to keep the temperature of the rotating container quasi-constant.

This process can be seen in analogy to the following process with the ideal gas. First consider the adiabatic expansion of the ideal gas. In this case the work done on the ideal gas is

$$dW = -pdV. \tag{7.4}$$

If we now want to do an isothermic expansion, then we can consider two infinitesimal steps: first, we do an infinitesimal expansion and second, we equilibrate with a reservoir. If these infinitesimal steps are repeated we have a process which is reversible and acts on a reservoir and the container of ideal gas during which we do the work $dW = -pdV$. but also exchange the heat $dU - pdV$ with the reservoir.

The described process acts not only on the reservoir and the black hole, but also on the container of gas. However, we assume that after the process the container is removed from the black hole and brought to its original state with a reversible work process. Therefore, the container does not change in entropy and we can ignore the container in the calculation of entropy.

Together with the reversible Penrose process, the isothermal process completes our search for a set of processes able to reversibly connect any two states in the black hole state space. Both processes are graphically represented in Figure 9.

8 Entropy

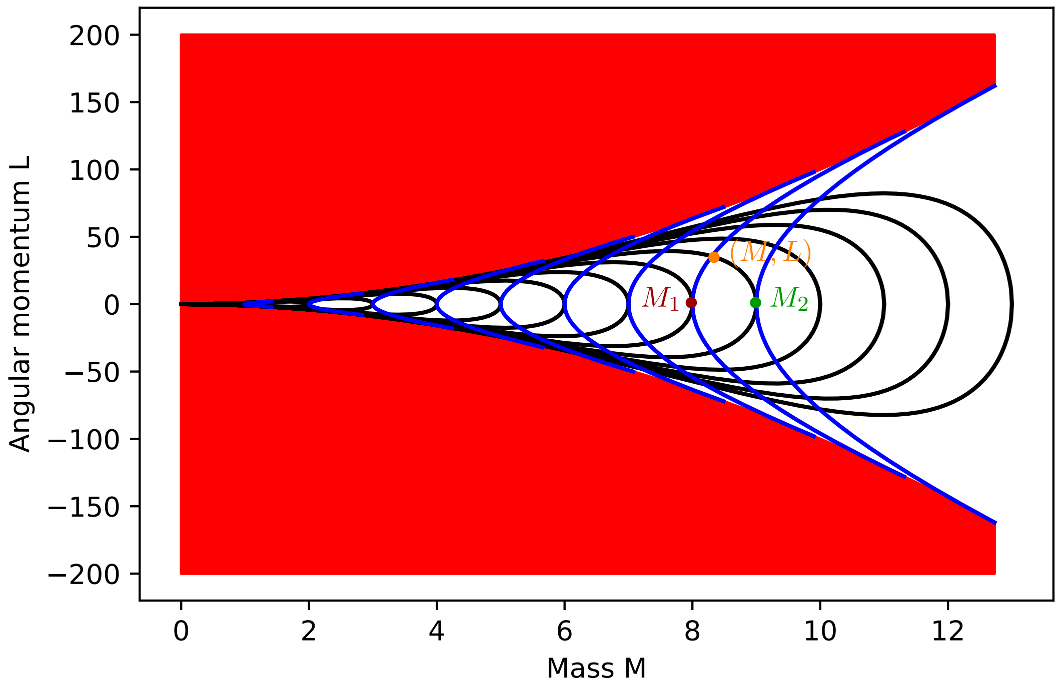

Figure 9: (L,M)-diagram of the black hole. The blue lines are the paths of the reversible Penrose processes and the black lines the paths of the isothermal processes.

Let us at first only consider non-charged black holes. For these black holes we have processes connecting any two states, the curves of which are depicted in Figure 9. The entropy is given by

$$\Delta S = S(\sigma_2) - S(\sigma_1) = \int_\gamma \frac{\delta Q_S}{T}. \tag{8.1}$$

The reversible Penrose process does not contribute to the entropy integral, as it is a work process on the black hole. Therefore, we can without loss of generality choose the start and end state to be $(M_1, 0)$ and $(M_2, 0)$; as all of the processes are reversible we can assume w.l.o.g. $M_1 \leq M_2$. If $M_1 > M_2$, the entropy difference is then the negative of the entropy difference calculated when the masses are interchanged. If $M_1 \neq M_2$, then these two states have different temperatures, otherwise we have already found a reversible process connecting the two states. To reach the end state with the isothermal process, we first need to apply a reversible Penrose process to the first state such that the resulting state (M, L) has the same temperature as the end state. Thus, this intermediary state has to fulfil the following equations

$$M^2 = M_1^2 + L^2/4M_1^2 \tag{8.2}$$

$$\frac{1}{8\pi M_2} = \frac{\sqrt{M^2 - L^2/M^2}}{4\pi M(M + \sqrt{M^2 - L^2/M^2})}. \tag{8.3}$$

To find the intermediary state, let us first eliminate the angular momentum from the above equation. The angular momentum and mass of the black hole during the reversible process connecting to the reservoir satisfy the equation

$$T = const. = \frac{\sqrt{M^2 - L^2/M^2}}{4\pi M(M + \sqrt{M^2 - L^2/M^2})}. \tag{8.4}$$

Solving this equation for $k = \sqrt{M^2 - L^2/M^2}$ we find

$$k = \frac{4T\pi M^2}{(1 - 4T\pi M)}. \tag{8.5}$$

Therefore, the following equation for L^2 holds

$$L^2 = M^4 - k^2 M^2 = M^4 - (\frac{4T\pi M^3}{(1 - 4T\pi M)})^2. \tag{8.6}$$

Eliminating L from Equation (8.3) with Equation (8.2) results in

$$\frac{1}{8\pi M_2} = \frac{\frac{2M_1^2}{M} - M}{8\pi M_1^2} \tag{8.7}$$

solving for M_1^2 we find

$$M_1^2 = -\frac{M^2}{2\left(\frac{M}{2M_2} - 1\right)}. \tag{8.8}$$

This equation can be solved for M and we find that

$$M = \frac{-\frac{M_1^2}{M_2} + \sqrt{(\frac{M_1^2}{M_2})^2 + 8M_1^2}}{2}, \tag{8.9}$$

which is always real and positive for any M_1 and M_2. Furthermore, this value of M can be reached reversibly by the reversible Penrose process as this expression is always smaller than $\sqrt{2M_1}$ and bigger than M_1. Furthermore, the state (M, L) has by construction the irreducible mass M_1 and is in the state space as by Equation (8.6) $|L| \leq M^2$. Thus, the reversible Penrose process and the isothermal process always intersect at one point. An example for $M_1, M_2, (M, L)$ can be seen in Figure 9.

Let us now focus on the differential work divided by the temperature during the process. It is convenient for the integration to express the differential work in therms of the change of mass during the process

$$\frac{\delta W_S}{T} = \frac{\Omega dL}{T} = \frac{24T^2\pi^2M^2 - 12T\pi M + 1}{T(4T\pi M - 1)^2}dM = \frac{3}{2T} - \frac{1}{2T(4T\pi M - 1)^2}dM. \tag{8.10}$$

where we used Equation (8.6) for the second equality.

With these results the entropy can be calculated

$$\begin{aligned}
\Delta S &= S(\sigma_2) - S(\sigma_1) \\
&= \int_\gamma \frac{\delta Q_S}{T} = \int \frac{dU_S - \delta W_S}{T} \\
&= (M - M_2)/T - \int_L^{L_2} \frac{\Omega dL}{T} \\
&= (M - M_2)/T - \int_M^{M_2} \left(\frac{3}{2T} - \frac{1}{2T(4T\pi M - 1)^2}\right) dM \\
&= (M - M_2)/T - \left(\frac{3}{2T}(M - M_2) - \frac{1}{2T}(M - M_2) + \frac{2M_2^2\pi}{4T\pi M_2 - 1} - \frac{2M^2\pi}{4T\pi M - 1}\right) \\
&= -\frac{2M_2^2\pi}{4T\pi M_2 - 1} + \frac{2M^2\pi}{4T\pi M - 1} \\
&= -\frac{2M_2^2\pi}{1/2 - 1} + \frac{2M^2\pi}{\frac{M}{2M_2} - 1} \\
&= 4\pi(M_2^2 - M_1^2) = (A_2 - A_1)/4.
\end{aligned} \tag{8.11}$$

Therefore the entropy of a state with area $A = 4\pi r_s^2 = 16\pi M_{irr}^2$ is

$$S = A/4 + const. \tag{8.12}$$

As can be seen in Figure 9, the reversible processes we used have the same derivative at $L = 0$. Therefore, we are not guaranteed to have a differentiable entropy. However, we can still calculate the derivative. For a general state (M, L) the entropy is given by

$$S(M, L) = 2\pi M(M + \sqrt{M^2 - L^2/M^2}) + const. \tag{8.13}$$

and has the partial derivatives

$$\frac{\partial S}{\partial M} = 4\pi(M + \frac{M^2}{\sqrt{M^2 - L^2/M^2}}) \tag{8.14}$$

$$\frac{\partial S}{\partial L} = -\pi\frac{2L}{\sqrt{M^4 - L^2}}. \tag{8.15}$$

These derivatives are continuous and defined for all (M, L) with $M > 0$ and $M^2 - L^2/M^2 > 0$. Thus, from theorem 9.21 in [26] it follows that the differential of S exists on the restricted part of the state space.

8.1 Charged black holes

Any state (M, L, Q) can be transferred via the generalized Penrose process to a state of the from $(M_{irr}, 0, 0)$, with M_{irr} as implicitly defined in Equation (4.10). As the reversible Penrose process does not change the entropy of the black hole, the entropy or the entropy difference between two such states can be calculated in the same way as above. Thus, the entropy for a state (M, L, Q) is given by

$$S(M, L, Q) = 4\pi M_{irr}^2 + const. = \pi((M + \sqrt{M^2 - Q^2 - L^2/M^2})^2 + L^2/M^2) + const. \quad (8.16)$$

There is a precaution that we need to consider with the Penrose process for the entropy calculation. As argued by Bekenstein [27] the reversible Penrose process is not realistic, because it requires ideal point particles, and as soon as the particles have some non-zero diameter, the process is no longer reversible. However, many processes used also for the ideal gas require some idealization where only infinitely slow processes are reversible.

9 Temperature

In the thermodynamic framework we use it is possible to define temperature of certain systems, when considering more than just energy we can also define the temperature of different conserved quantities. In the case of the black hole, of the angular momentum and the charge. These temperatures are then given by derivative of the internal conserved quantity with regards to the entropy, as it is usually done in the case of the energy temperature.

In this section, we will only consider the space where $M^2 - L^2/M^2 - Q^2 > 0$, as in this region the entropy is differentiable. Furthermore, we can also not use the temperature definition, as the entropy is not differentiable for black holes with $M^2 - L^2/M^2 - Q^2 = 0$.

Energy temperature Conveniently, the internal energy of a black hole is just its mass, thus

$$\begin{aligned} T &= \frac{\partial M}{\partial S} = \left(\frac{\partial S}{\partial M}\right)^{-1} \\ &= \frac{1}{\pi}\left(4M + \frac{2(2M^2 - Q^2)}{\sqrt{M^2 - Q^2 - L^2/M^2}}\right)^{-1} \\ &= \frac{1}{2\pi}\frac{\sqrt{M^2 - L^2/M^2 - Q^2}}{2M(M + \sqrt{M^2 - L^2/M^2 - Q^2}) - Q^2}. \end{aligned} \quad (9.1)$$

This is exactly the temperature of the Hawking radiation we discussed before.

Angular momentum temperature Luckily, the internal angular momentum of a black hole is just its angular momentum, thus

$$
\begin{aligned}
T_L = \frac{\partial L}{\partial S} &= (\frac{\partial S}{\partial L})^{-1} \\
&= (-\pi \frac{2L}{\sqrt{M^2(M^2 - Q^2) - L^2}})^{-1} \\
&= -\frac{\sqrt{M^2(M^2 - Q^2) - L^2}}{2\pi L} \\
&= -\frac{T}{\Omega}.
\end{aligned}
\tag{9.2}
$$

Interestingly, the variable Ω of a black hole has three of the same properties as the angular velocity of the spinning ideal gas: first, the black hole inside a rotating ideal gas does not change if the angular velocity of the rotating gas is also Ω. Second, the work done on the black hole during the reversible process is given by $-\Omega dL$, same as for a rotating ideal gas. Third, the angular temperature has the same form as for a rotating ideal gas.

Charge temperature Fortunately, the internal charge of a black hole is just its charge, thus

$$
\begin{aligned}
T_Q = \frac{\partial Q}{\partial S} &= (\frac{\partial S}{\partial Q})^{-1} \\
&= (-\pi \frac{2M^2 Q}{\sqrt{M^2(M^2 - Q^2) - L^2}} - 2Q)^{-1} \\
&= -\frac{\sqrt{M^2(M^2 - Q^2) - L^2}}{2\pi(\sqrt{M^2(M^2 - Q^2) - L^2} + M^2)} \\
&= -\frac{T}{\Phi}
\end{aligned}
\tag{9.3}
$$

with

$$
\Phi = \frac{Q(M + \sqrt{M^2 - Q^2 - L^2/M^2})}{2M(M + \sqrt{M^2 - Q^2 - L^2/M^2}) - Q^2}. \tag{9.4}
$$

Physically, Φ can be seen as the electrostatic potential of the black hole.

9.1 Phenomenological temperatures and Hawking radiation

We have calculated these temperatures from the entropy. The question then arises: is it possible to link these temperatures to the Hawking radiation? To this end, let us consider the Hawking radiation of a charged and rotating black hole (for details see [28]). In this case one finds the number of particles emitted in the mode ω, j, n, l, m given by

$$
p_{jnlm}(\omega) = \frac{\Gamma_{jnlm}}{e^{\frac{(\omega - m\Omega - e\Phi)}{T_{bh}}} \pm 1} \tag{9.5}
$$

where Γ_{jnlm} is the grey body factor, e the charge of the field of which the wave packet is considered, and m the angular momentum in the direction of the black hole. The different signs in the numerator are for boson ($-$) and fermion ($+$) fields. Up to the grey body factor, this is the thermal distribution with the above temperatures for energy, charge and angular momentum. Thus, the angular momentum, and charge temperatures calculated above are not just the temperatures we would phenomenologically assign, but they are also reproduced in the Hawking radiation of a black hole.

10 Bekenstein's entropy derivation

In this section we present a second information theoretic but also more informal way by Bekenstein [29] to derive the entropy of a black hole. The idea of his derivation is to connect entropy to information, by equating the lost information in a black hole with the negative of the entropy change

$$\Delta I = -\Delta S. \tag{10.1}$$

In general, we want that entropy of a black hole can only increase. Thus, Bekenstein consideres entropy to be a monotonically increasing function f of the rationalized area $\alpha = A/4\pi$, which by Hawking's area theorem [21] can only increase under classical processes. Thus,

$$S_{bh}(\alpha) = f(\alpha). \tag{10.2}$$

A first example of f that can be considered is $f = \gamma\alpha^{1/2}$. However, this choice of f leads to a problem when we consider two colliding black holes. When two black holes collide the mass of the resulting black hole is in general equal or smaller[2] than the sum of the masses of the colliding black holes. But in this case the total entropy decreases, which we do not want. A second example of f is $f = \gamma\alpha$, in this case the above problem does not appear. Therefore we choose this functional dependence. Let us try to determine γ, the units of γ need to be of the form $\frac{1}{\text{length}^2}$[3]. A constant with the correct units is $\hbar^{-1}$, so we set $f = \eta\hbar^{-1}\alpha$. To find the dimensionless constant η, we consider the process where we throw an elementary particle into the black hole, the minimal entropy increase in this case is

$$(\Delta S)_{min} = \ln(2), \tag{10.3}$$

because the information weather the particle exists or not is lost.

The minimal increase of α under such a process is $\Delta\alpha = 2\mu b$, where μ is the mass of the particle and b its radius. We can bound b from below with the Schwarzschild radius (2μ) and the Compton wave length ($\hbar\mu^{-1}$) of the particle. Combining these two bounds we can conclude that $(\Delta\alpha)_{min} = (2\mu b)_{min} > 2\hbar$. So the minimal increase of the entropy during this process is also $(\Delta S)_{min} = 2\hbar\frac{df}{d\alpha}$. Equating this to the previously found minimal entropy change we find

$$S_{bh}(\alpha) = ln(2)/2\alpha. \tag{10.4}$$

The derivation of the prefactor is quite hand wavy and in particular relying on the assumption that the minimal size of the particle is its Compton wavelength, so it is expected that the prefactor differs from the prefactor obtained in the previous calculation. However the scaling that was found by Bernstein with this derivation is correct.

[2]Some of the mass could also be going into gravitational radiaton
[3]Here we use the units $c = G = 1$

11 Geroch process

Let us consider a process by Geroch[4] which seemingly violates the second law. In the following we will consider non-charged and non-rotating black holes. The Geroch process considers a box filled with radiation of energy μ at infinity. This mass is slowly lowered on a string to the black hole until it is directly at the event horizon. At the event horizon the red shift is so large that when the content of the box are dropped in the black hole, it does not gain any mass. Then the empty box is pulled up again. The work W which is done on the box and the black hole is given by

$$W_{\text{BH, Box}}(p_{Geroch}) = -\mu \tag{11.1}$$

and the initial and final states of the black hole and reservoir are given by

$$\lfloor p_{Geroch} \rfloor_{BH} = M,\ \lceil p_{Geroch} \rceil_{BH} = M \tag{11.2}$$

$$\lfloor p_{Geroch} \rfloor_{R} = U_0,\ \lceil p_{Geroch} \rceil_{R} = U_0 - \mu. \tag{11.3}$$

In summary, the black hole state is not changed and we can at infinity fill up the box again with radiation from a reservoir. This means that the process is cyclic on the black hole and the box. Furthermore, the work done on these systems is negative. Therefore, the process violates the second law.

11.1 Resolutions

Let us abstract this process such that we only consider the initial and final states and the work costs of all involved systems. To no longer violate the second law, let us change the process such that it becomes acyclic. We denote the change in mass of the black hole after the process by $\chi\mu$ where χ is chosen such that $\chi\mu$ is exactly the black hole mass change. As the mass of the black hole is also its internal energy, by conservation of energy it has to hold that the work of the modified process changes to

$$W_{\text{BH, Box}}(p_{Geroch}) = -(1-\chi)\mu. \tag{11.4}$$

Even though when the process is no longer cyclic it does not violate the second law directly. However, it could still violate the second law indirectly by, for example, violating theorems derived from the second law.

First, we consider the entropy theorem, which states that the total entropy of the initial state must be lower than the entropy of the final state. Therefore, it must hold that the entropy of the box's content S_{box} and the entropy change of the black hole must fulfil

$$4\pi((M+\chi\mu)^2 - M^2) - S_{box} \geq 0. \tag{11.5}$$

Because the box is filled with radiation from a reservoir, it has to hold that $S_{box} \geq \frac{\mu}{T_{res}}$, as the entropy change of a reservoir when the energy E is extracted is given by $-\frac{E}{T}$. If we assume that the energy in the box μ is small, we can reduce the above inequality to

$$8\pi M\chi \geq \frac{S_{box}}{\mu} \geq \frac{1}{T_{res}}. \tag{11.6}$$

[4]Presented at a Colloquium at Princeton University in December 1971. A description of this process can be found in [29].

Second, we consider the implications of Carnot's theorem. To apply Carnot's theorem we need to make the process cyclic. We can make the process cyclic by applying the process we used to calculate the entropy difference between two states $(M_1, 0, 0)$ and $(M_2, 0, 0)$. The heat extracted from the second reservoir of temperature $T_2 = \frac{1}{8\pi(M+\mu)}$ during this process is given by $Q_{R_2}(p) = \chi\mu/2 + \frac{M\chi}{2(M+\chi\mu)}$. The heat extracted from the original reservoir is given by $Q_{R_1}(p) = -\mu$. From Carnot's theorem it must hold that

$$-\frac{Q_{R_1}(p)}{Q_{R_2}(p)} = \left(\frac{\frac{\chi\mu}{2} + \frac{M\chi\mu}{2(M+\chi\mu)}}{\mu}\right)^{-1} \leq \frac{T_{res}}{T_2} = 8\pi(M+\mu)T_{res}. \tag{11.7}$$

This equation can be simplified to

$$\chi(8\pi M + \chi\mu) \geq 1/T_{res}, \tag{11.8}$$

which is the same equation as obtained by application of the second law.

11.1.1 Process changes

Now that we have considered how the process needs to change in a thermodynamic way, let us consider how this maps to the physical changes of the process. The first change that we consider was proposed was by Bekenstein in [29]. He considers the box to be spherical and argues that to contain radiation the box to have a finite radius b. In this case, the box can only be brought to a distance b within the black hole horizon. Therefore, the mass of the black hole must increase by at least

$$\chi\mu = 2\pi\mu b T_{bh}(M) = \frac{\mu b}{4M}. \tag{11.9}$$

For a derivation of this equation see [29]. If we use this χ in Equation (11.6), we find

$$2\pi b \geq \frac{S_{box}}{\mu} \geq \frac{1}{T_{res}}. \tag{11.10}$$

Interestingly, this resulting equation no longer contains any reference to the black hole and puts the size of a system, its internal energy, and entropy in relation with each other. This result was also derived by Bekenstein in [30] and is known as the Bekenstein-Casini bound.

A second approach also considering the radiation pressure of the Hawking radiation was done by Wald and Unruh in [31]. They argue that the energy change of the black hole is minimized when a box filled with thermal radiation is lowered to the point where the box is buoyant, i.e. the tension on the string is zero. This point is reached when the energy of the box is equal to the energy of the displaced Hawking radiation. Then it can be argued that the entropy change of the black hole is given by the entropy of the displaced radiation. Furthermore, the entropy inside the box cannot be larger than the entropy of the thermal radiation from the black hole, as it maximises entropy for a fixed volume and entropy. This resolution has one problem as it does not address the case of a collapse, which was pointed out by Susskind [32]. If a body with mass M and entropy S such that $S \geq 4\pi M^2$ collapses, then the second law is also violated.

Let us briefly discuss where the bounds we found above apply. We have argued that if we assume the postulates and the process descriptions to be true, then the Bekenstein-Casini bound follows. However, we have not proven that these postulates need to hold. Furthermore, we have only considered a spherical box being lowered to the black hole, and as pointed out by Unruh and Wald [31] Bekenstein's argument breaks down if the box is no longer spherical but shaped as a thin cuboid. However, Casini [33] has been able to rigorously state and proof the Bekenstein-Casini bound within quantum field theory.

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