Discrimination of thermal states

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Abstract

We study several variations of the question of minimum-error discrimination of thermal states. Except of providing the optimal values for the probability error we also characterize the optimal measurements. For the case of fixed Hamiltonian we show that for a general discrimination problem the optimal measurement is the measurement in the energy basis of the Hamiltonian. We identified critical temperature determining whether the given temperature is best distinguishable from thermal state of very high, or very low temperatures. Further, we investigate the decision problem whether the thermal state is above, or below some threshold value of the temperature. Also in this case the optimal minimum-error measurement is the measurement in the energy basis. This is no longer the case once the states to be discriminated have different Hamiltonians. We analyze specific situation when the temperature is fixed, but the Hamiltonians are different. For the considered case we show the optimal measurement is independent of the fixed temperature and also of the strength of the interaction.

Introduction

For quantum systems the temperature is assigned to density operators of the form

$$\rho_\beta(H) = \frac{e^{-\beta H}}{\mathrm{tr} e^{-\beta H}}, \quad (1)$$

For a general binary minimum-error discrimination the optimal value of success probability is known [?] to be given by the Helstrom formula

$$p_{\mathrm{error}} = \frac{1}{2}\left(1 - \frac{1}{2}||\varrho_1 - \varrho_2||_1\right), \quad (2)$$

Goals

Our goal is simple: evaluate the minimum error probability for thermal states and analyse the results. We are interested to understand when the difference of the temperatures matters, i.e. increases the distinguishability, and how the parameters of the Hamiltonian affects the distinguishability. Is it easier to distinguish larger, or smaller temperatures? Does the strength of the interaction increases the distinguishability, or not?

1 Binary case with fixed Hamiltonian

$$||\rho_1 - \rho_2||_1 = \sum_j \left| \frac{\exp(-\beta_1 E_j)}{Z_1} - \frac{\exp(-\beta_2 E_j)}{Z_2} \right|, \quad (3)$$

where $Z_1 = \sum_j \exp(-\beta_1 E_j)$ and $Z_2 = \sum_j \exp(-\beta_2 E_j)$. Using the associated energy distributions $w_{1j} = w_{\beta_1}(E_j)$ and $w_{2j} = w_{\beta_2}(E_j)$ the minimum error probability equals

$$p_{\mathrm{error}} = \frac{1}{2}\left(1 - \frac{1}{2}\sum_j |w_{1j} - w_{2j}|\right). \quad (4)$$

Measurement in energy basis is optimal

Let us now consider that the measurement performed is the energy measurement, i.e. outcomes E_j associated with projectors Π_j, hence, $w_{xj} = \mathrm{tr}[\varrho_x \Pi_j] = (1/Z_x)\exp(-\beta_x E_j)$. Inserting these numbers into the above formula we observe that the obtained expression coincides with the optimal value for minimum error discrimination of two thermal states.

Case study: qubit

The general Hamiltonian has the form $H = aI + \vec{\alpha}\cdot\vec{\sigma}$. Let us introduce the quantity $\alpha = ||\vec{\alpha}|| > 0$ and operator $S = \vec{\alpha}\cdot\vec{\sigma} = |\varphi_+\rangle\langle\varphi_+| - |\varphi_-\rangle\langle\varphi_-|$, where $|\varphi_\pm\rangle$ are eigenvectors of H and S. The eigenvalues of $H = aI + \alpha S$ reads $E_\pm = a \pm \alpha$, thus, we obtain energies $E_0 = E_-$ for the ground state and $E_1 = E_+$ for the excited one. Direct calculation gives

$$p_{\mathrm{error}} = \frac{1}{2}\left(1 - \frac{1}{2}\frac{|\sinh\alpha(\beta_1 - \beta_2)|}{2\cosh(\beta_1\alpha)\cosh(\beta_2\alpha)}\right). \quad (5)$$

$$\Pi_0 = \frac{1}{2}(I - S) = |\varphi_-\rangle\langle\varphi_-| \quad (6)$$
$$\Pi_1 = \frac{1}{2}(I + S) = |\varphi_+\rangle\langle\varphi_+|$$

Further, without loss of generality we may assume $\beta_1 > \beta_2$ (i.e. $T_1 < T_2$). Then

$$w_{10} = \mathrm{tr}[\varrho_1\Pi_0] = \frac{e^{\beta_1\alpha}}{Z_1} > \frac{e^{\beta_2\alpha}}{Z_2} = \mathrm{tr}[\varrho_2\Pi_0] = w_{20}$$
$$w_{11} = \mathrm{tr}[\varrho_1\Pi_1] = \frac{e^{-\beta_1\alpha}}{Z_1} < \frac{e^{-\beta_2\alpha}}{Z_2} = \mathrm{tr}[\varrho_2\Pi_1] = w_{21},$$

where $Z_j = 2\cosh\beta_j\alpha$. Therefore, if the ground energy is recorded we conclude the smaller of the temperatures (T_1), whereas, if excited energy is observed, then the state of the larger temperature (T_2) is identified.

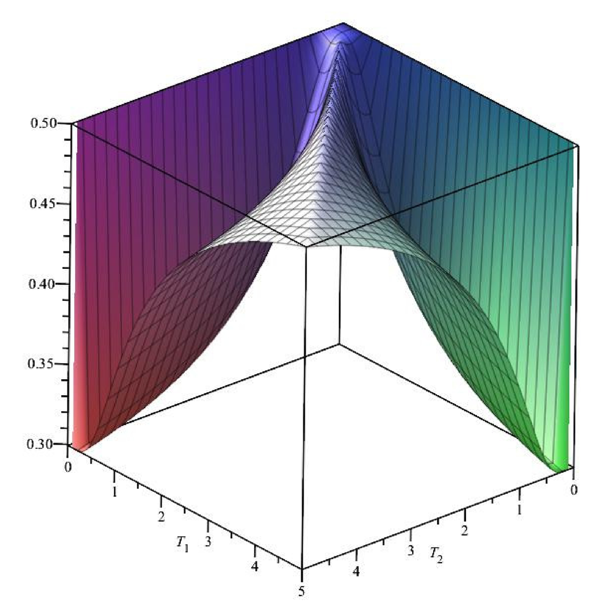

Figure 1: 3D plot of probability of error between two satets ϱ_1 and ϱ_2 in terms of T_1 and T_2 in the presence of same Hamiltonian $H = \sigma_z$.

Dependence on the difference of temperatures

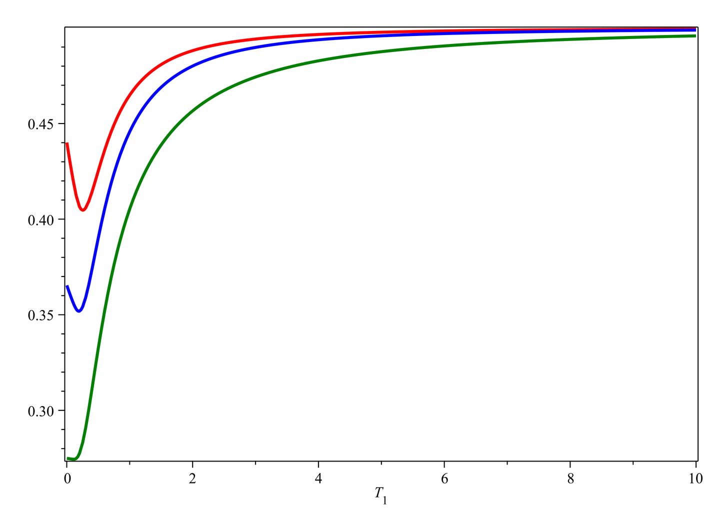

Figure 2: Probability of error for fixed values of $\Delta T = 0.5$ (Red line),$\Delta T = 1$ (Blue Line) and $\Delta T = 5$ (Green Line). All plots are depicted for $\alpha = 0.5$.

Qudits: Finite dimensional linear harmonic oscillator

$$H = E_0\boldsymbol{I} + \alpha\sum_{j=1}^{d-1} j|j\rangle\langle j|. \quad (7)$$

$$q_0 = \lim_{T_1\to 0}||\varrho_1 - \varrho_2||, \quad q_\infty = \lim_{T_1\to\infty}||\varrho_1 - \varrho_2|| \quad (8)$$

If $q_\infty \geq q_0$, then T_2 is best discriminated with temperatures T_1 from the area of very low temperatures. Otherwise, T_2 is best discriminated with higher temperatures T_1. The identity $q_0 = q_\infty$ identifies the critical value of temperature T^* determining whether the minimum of error probability of distinguishing T_2 and T_1 is achieved for larger, or small temperatures T_1.

The condition $q_0 = q_\infty$ implies

$$\frac{d-1}{d+1} = \sum_{j=1}^{d-1}\exp\left(-\frac{j\alpha}{T^*}\right) = \frac{\exp(-\frac{\alpha}{T^*}) - \exp(-\frac{N\alpha}{T^*})}{1 - \exp(-\frac{\alpha}{T^*})}. \quad (9)$$

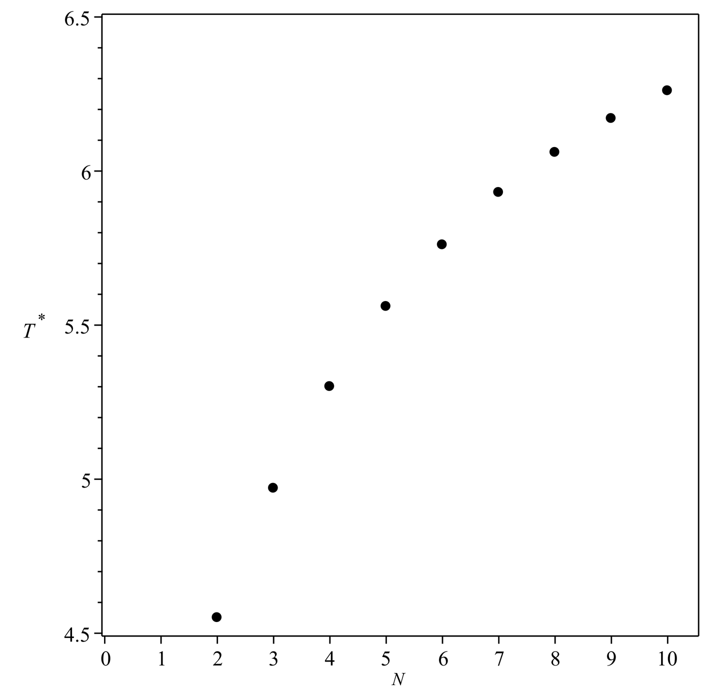

Figure 3: Critical Temperatures for different dimensions for fixed $\alpha = 5$.

For the simplest cases we can evaluate the values also analytically

$$T^* = \frac{\alpha}{\ln(3)} \quad \text{for } d = 2$$
$$T^* = -\frac{\alpha}{\ln(\frac{2}{\sqrt{3}-1})} \quad \text{for } d = 3. \quad (10)$$

2 Case of multiple temperatures

Theorem 1. *Consider a set of mutually commuting states $\varrho_1, \ldots, \varrho_N$ with apriori probabilities $\eta_1, \ldots, \eta_N$. Let us denote by Π_k the eigenprojectors of ϱ_j, thus, $\varrho_j = \sum_k w_{jk}\Pi_k$ for all j and $w_{jk} = \mathrm{tr}[\varrho_j\Pi_k]$. Define the index set I_j composed of indexes k for which $\eta_j\mathrm{tr}[\varrho_j\Pi_k]$ is maximized by ϱ_j. Then the optimal minimum-error discrimination measurement is composed of projectors $F_j = \sum_{k\in I_j}\Pi_k$ identifying the conclusion ϱ_j. The probability of success equals*

$$p_{\mathrm{success}} = \sum_k \mathrm{tr}[\Pi_k]\max\{\mathrm{tr}[\eta_1\varrho_1\Pi_k], \ldots, \mathrm{tr}[\eta_N\varrho_N\Pi_k]\}.$$

Qubits

$$\varrho_j = \frac{e^{\alpha\beta_j}\Pi_0 + e^{-\alpha\beta_j}\Pi_1}{2\cosh[\alpha\beta_j]}. \quad (11)$$

Discrimination of N thermal states of qubits concludes with nonzero probability only the states with minimal and maximal temperatures, i.e.

$$p_{\mathrm{error}} = 1 - \frac{1}{N}(\mathrm{tr}[\varrho_1\Pi_0] + \mathrm{tr}[\varrho_N\Pi_1])$$
$$= 1 - \frac{1}{N}(\mathrm{tr}[(\varrho_1 - \varrho_N)\Pi_0] + 1)$$
$$= \frac{N-1}{N} + \frac{1}{2N}\left(\frac{\sinh\alpha(\beta_1 - \beta_N)}{\cosh\alpha\beta_1\cosh\alpha\beta_N}\right) \quad (12)$$

Temperature threshold

consider the source is producing one of increasingly-ordered thermal states $\varrho_1, \ldots, \varrho_N$, the goal is decide only whether the temperature is above, or below some specified (threshold) value T_c. Assuming all the states are equally likely we may introduce density operators

$$\varrho_- = \frac{1}{N_-}\sum_{j\in S_-}\varrho_j, \qquad \varrho_+ = \frac{1}{N_+}\sum_{j\in S_+}\varrho_j,$$

where $N_- + N_+ = N$ and $N_\pm$ labels the number of thermal states below and above the specified temperature, respectively. The discrimination problem is reduced to discrimination of these averages of thermal states apriori distributed as $q_\pm = N_\pm/N$. The state $\varrho_\pm$ are themselves not thermal states (except of the qubit case), however, they are still commuting and diagonal in the energy basis. Consider the case of qubit states.

$$\varrho_\pm = \frac{1}{2}\left(I - \tanh\left(\frac{\alpha}{T_\pm}\right)\sigma_z\right), \quad (13)$$

$$T_\pm = \frac{1}{\alpha}\tanh^{-1}\left(\frac{\sum_{j\in S_\pm}\tanh(\frac{\alpha}{T_j})}{N_\pm}\right). \quad (14)$$

The optimal measurement (Theorem 1) consists of projectors onto eigenvectors of the Hamiltonian

$$\pi_1 = |1\rangle\langle 1|,$$
$$\pi_2 = |0\rangle\langle 0|. \quad (15)$$

The conclusion made (above/below threshhold), when the outcome π_j is registered, follows from the comparison of values $q_\pm tr(\varrho_\pm\pi_j)$.

3 Thermal States with different Hamiltonians and fixed temperature

$$p_{\mathrm{error}} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\left|\tanh\left(\frac{B}{T}\right)\right|\sqrt{1 - \vec{b}_1\cdot\vec{b}_2}\right). \quad (16)$$

Expressing $\vec{v}_j$ via $\vec{b}_j$ and assuming, for simplicity, that $\eta_1 = \eta_2 = 1/2$ it follows $\pi_\pm = \frac{1}{2}(I \pm \frac{\vec{b}_1 - \vec{b}_2}{||\vec{b}_1 - \vec{b}_2||}\cdot\vec{\sigma})$. As a result we see that optimal measurement is measuring the spin along the direction $\vec{b}_1 - \vec{b}_2$. Moreover, let us stress it is independent of the temperature T and also of the strength of the magnetic field B.

Conclusions

- The measurement in energy basis is the optimal one for discrimination of pairs of thermal states with the same Hamiltonian.
- Finding the critical temperature T_* such that by comparing a temperature T with T_* one can find if it can be better discriminated with higher or lower temperature.
- Generalizing some of the results to the case of a d dimensional Hamiltonian and formulation the optimal discrimination for a general set of mutually commuting states.
- Identification problem for deciding whether the temperature of thermal states is above, or below some threshold value of the temperature T_c.
- Extension to non-commutative case: For unitarily related and fixed temperature, the optimal discrimination measurement is independent of the (fixed) temperature and the interaction strength.

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