Fluctuation-dissipation relations for thermodynamic distillation processes

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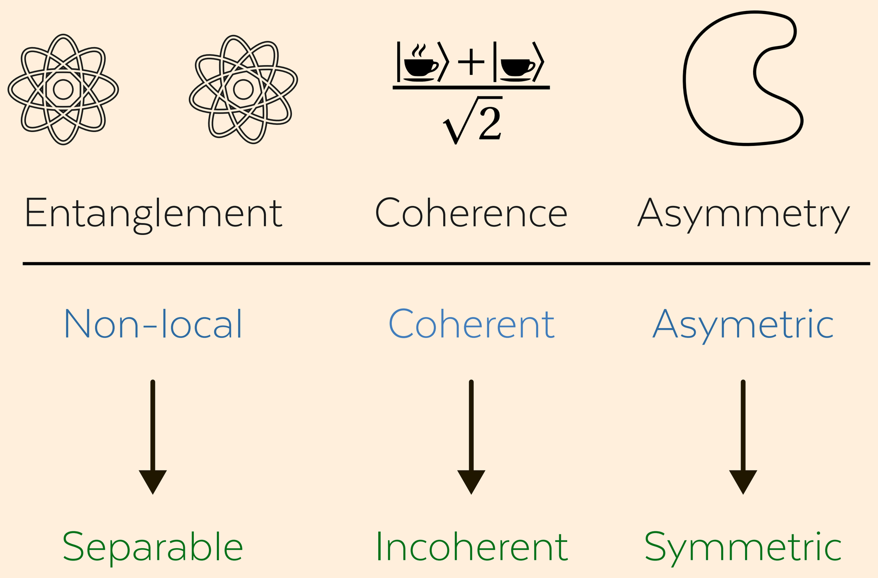
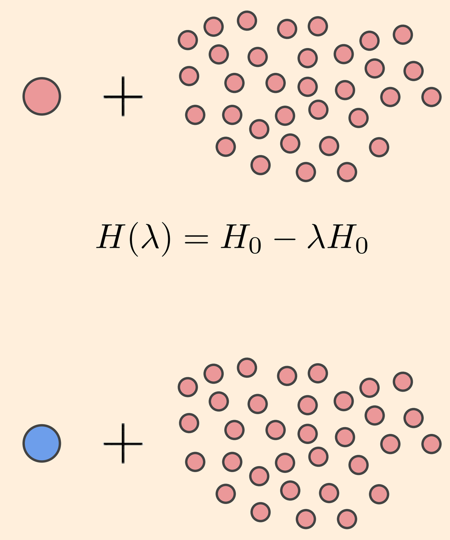
Background

Goal: study resource dissipation and characterise optimal state transformation protocols that minimise dissipation

Framework: ordering among quantum states



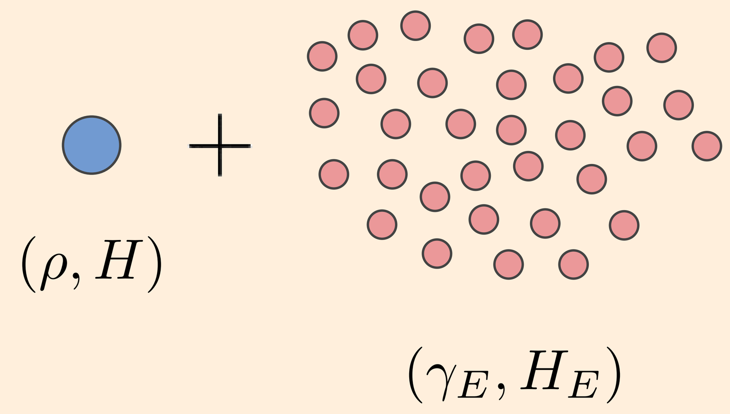


$H(\lambda) = H_0 - \lambda H_0$

$W^{\text{diss}} \simeq \beta\lambda[\langle H^2\rangle_0 - \langle H\rangle_0^2]$

Setting the scene

Indentifying the set of **thermodynamically-free states**



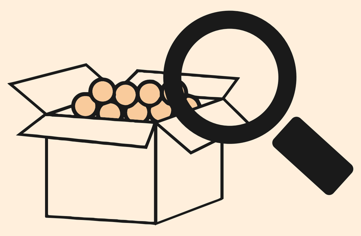
$$\Longrightarrow \gamma = \frac{e^{-\beta H}}{Z} \quad , \quad Z = \text{Tr}(e^{-\beta H})$$

Thermodynamic transformations are modelled by **thermal operations**

$$\mathcal{E}(\rho) = \text{Tr}_E(U(\rho\otimes\gamma_E)U^\dagger) \quad \text{with} \quad [U, H\otimes\mathbb{1}_E + \mathbb{1}_E\otimes H_E] = 0$$

Energy-conserving interaction

Thermodynamic **monotone** $\phi : \mathcal{S}_d \to \mathbb{R}_+ \cup \{0\}$



i. $\phi(\mathcal{E}(\rho)) \leq \phi(\rho)$

ii. $\phi(\gamma) = 0$

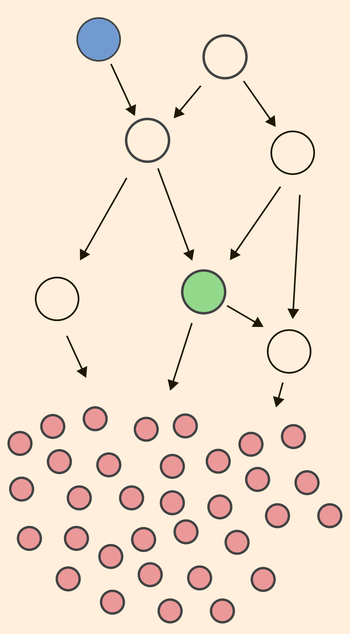
$$D(\rho\|\gamma) = \text{Tr}(\rho(\log\rho - \log\gamma))$$

Generalised **free energy**

Thermodynamic distillation process

For initial state ρ, target state σ, thermal bath $\beta \Longrightarrow \mathcal{E}(\rho) = \sigma$

out of equilibrium



equilibrium

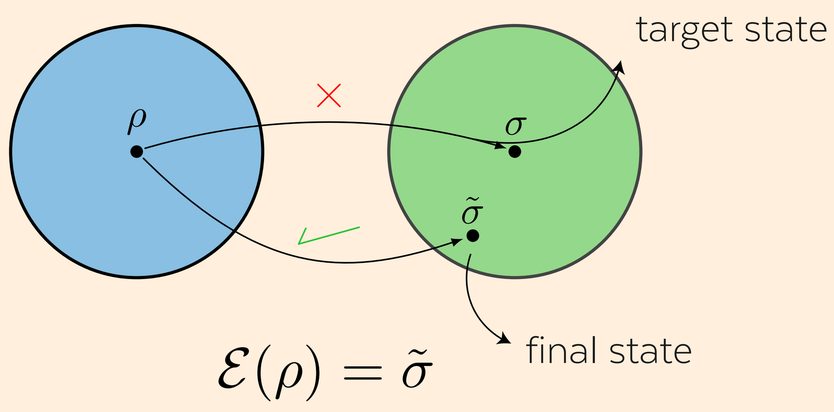
! General answer not known beyond the simplest qubit case

! For **energy-incoherent** states

$$\mathcal{E}(\rho) = \sigma : \mathbf{p} \succ^\beta \mathbf{q}$$

ϵ -approximate **interconversion** problem:

$\sigma \approx_\epsilon \tilde{\sigma}$ means $1 - F(\sigma, \tilde{\sigma}) \leq \epsilon$



$\mathcal{E}(\rho) = \tilde{\sigma}$

ϵ -approximate **thermodynamic distillation process**: $(\rho, H) \xrightarrow{\mathcal{E}} (\tilde{\rho}, \tilde{H})$

with $\tilde{\rho} = \bigotimes_{m=1}^{\tilde{N}} |\tilde{E}_{k_n}^{(n)}\rangle\langle\tilde{E}_{k_n}^{(n)}|$

Eigenstate of $|\tilde{E}_{k_n}^{(n)}\rangle$ correspoding to energy $\tilde{E}_{k_n}^{(n)}$

$$\Delta F^N := \frac{1}{\beta}\left(\sum_{n=1}^N D(\rho_n^N\|\gamma_n^N) - D(\tilde{\rho}^N\|\tilde{\gamma}^N)\right)$$

Free energy **change**

$$\sigma^2(F^N) := \frac{1}{\beta^2}\sum_{n=1}^N V(\rho_n^N\|\gamma_n^N)$$

Free energy **fluctuations**

Results

Theorem 1. For a distillation setting with energy incoherent initial states, the transformation error of the approximate distillation process in the asymptotic limit is given by

$$\lim_{N\to\infty} \epsilon_N = \lim_{N\to\infty} \Phi\left(-\frac{\Delta F^N}{\sigma(F^N)}\right)$$

Moreover, for any N there exist an approximate distillation process with the transformation error bounded by

$$\epsilon_N \leq \Phi\left(-\frac{\Delta F^N}{\sigma(F^N)}\right) + \frac{C\kappa^3(F^N)}{\sigma^3(F^N)}$$

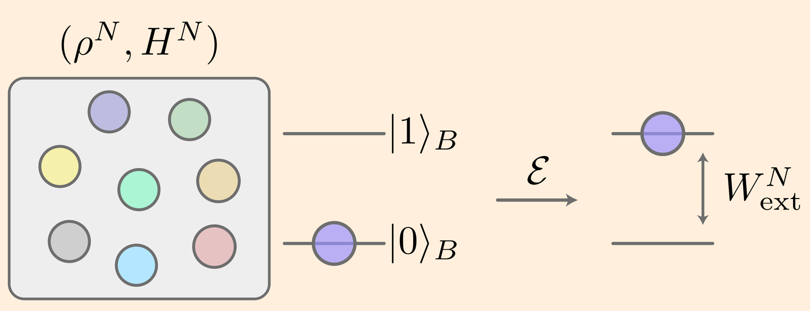
Theorem 2. For a distillation setting with N identical initial systems each in a pure state and described by the same Hamiltonian, the transformation error of the optimal approximate distillation process in the asymptotic limit is given by

$$\lim_{N\to\infty} \epsilon_N = \lim_{N\to\infty} \Phi\left(-\frac{\Delta F^N}{\sigma(F^N)}\right)$$

The amount of **free energy dissipated** in both settings satisfies:

$$F_{\text{diss}}^{\text{tot}} = a(\epsilon)\,\sigma^{\text{tot}}(F) \quad \text{with} \quad a(\epsilon) = -\Phi^{-1}(\epsilon)(1-\epsilon) + \frac{\exp\left(\frac{-(\Phi^{-1}(\epsilon))^2}{2}\right)}{\sqrt{2\pi}}$$

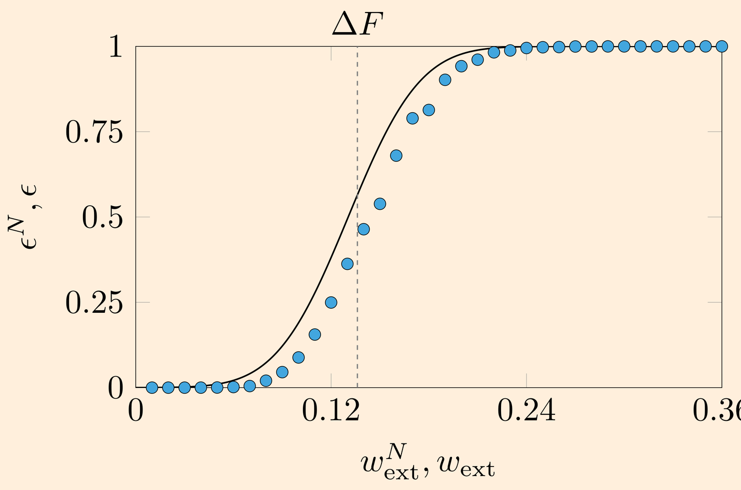
Aplications



Optimal work extraction

- *Theorem 1* tell us that the optimal transformation error for extracting the amount of work W_{ext}^N is:

$$W_{\text{ext}}^N \simeq F^N + \sigma(F^N)\Phi^{-1}(\epsilon)$$

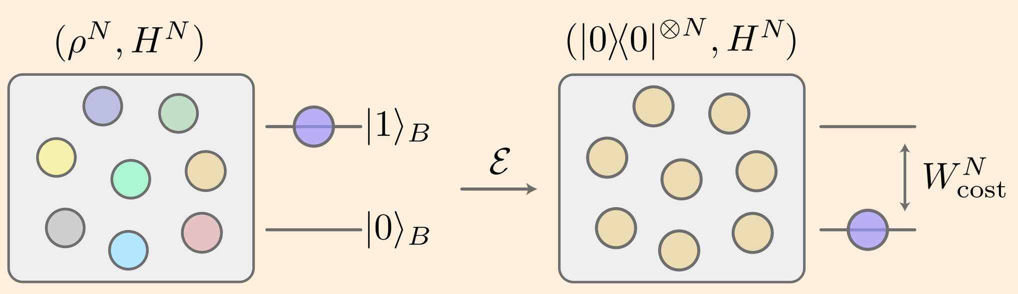


Initial system is composed of 100 two-level subsystems. The first 59 is prepared in the state p = (0.9, 0.1) with thermal state (0.6,0.4) and remaning is prepared in a state (0.7, 0.3) and thermal state (0.75, 0.25).

- From *Theorem 2*, the optimal amount of work extracted from N pure quantum systems up to second-order is given by

$$W_{\text{ext}} \simeq N\left(\langle H\rangle_\psi + \frac{\log Z}{\beta} + \frac{\langle H^2\rangle_\psi - \langle H\rangle_\psi^2}{\sqrt{N}}\Phi^{-1}(\epsilon)\right)$$

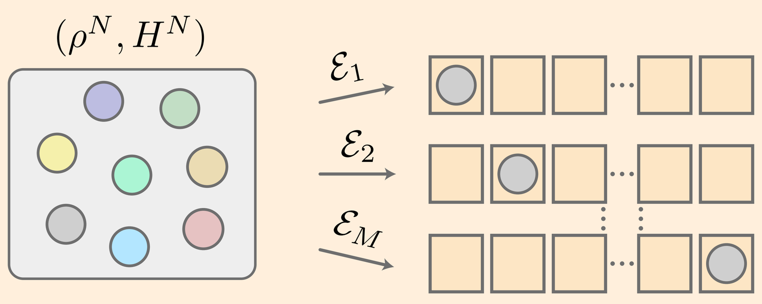
Optimal information erasure



applying *Theorem 1* yields:

$$W_{\text{cost}}^N \simeq \frac{S(\rho^N)}{\beta} - \sigma(F^N)\Phi^{-1}(\epsilon)$$

Optimal thermodynamically-free communication rate



The optimal number of messages that can be encoded into in a thermodynamically-free way:

$$R(\rho^{\otimes N}, \epsilon_{\text{d}}) \simeq D(\rho\|\gamma) + \frac{\sqrt{V(\rho\|\gamma)}}{\sqrt{N}}\Phi^{-1}(\epsilon_{\text{d}})$$

! This result is valid for either a pure or incoherent state!