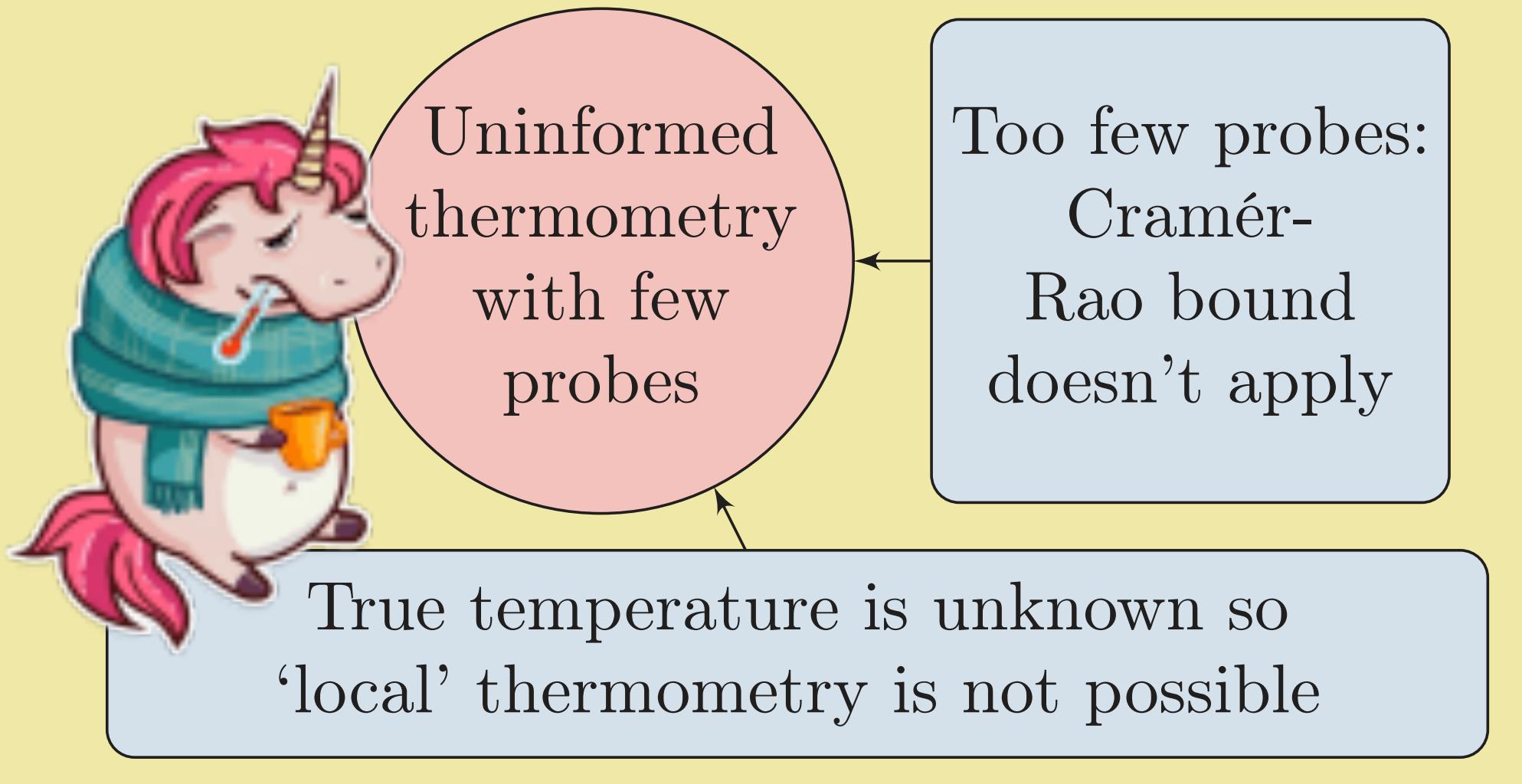
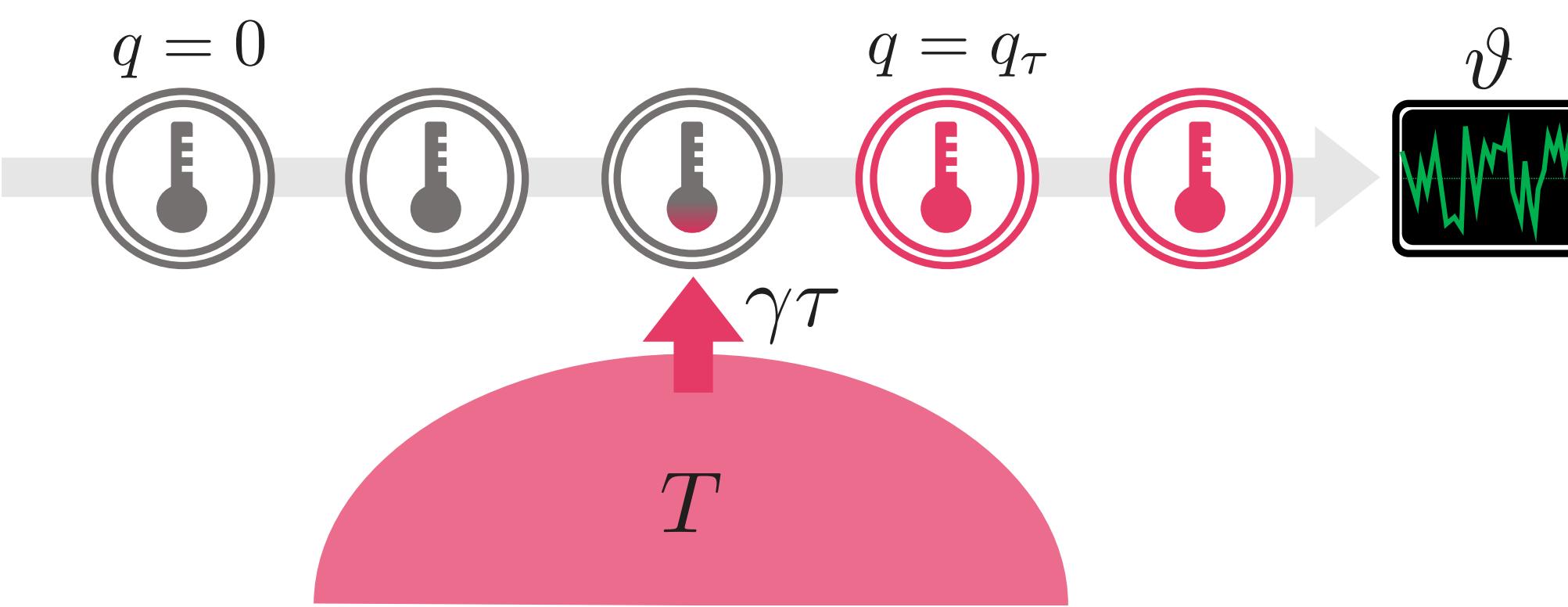


WHY BAYESIAN ESTIMATION?



QUBIT THERMOMETRY



N ground-state qubits exchange heat with a bosonic reservoir at temperature T and rate γ , each for a duration τ . The qubit ensemble is then in a mixed state with excitation probability:

$$q_\tau(T) = \frac{1 - e^{-\gamma\tau \coth(E/2k_B T)}}{1 + e^{E/k_B T}}.$$

The experimenter infers the temperature from the measured number of excitations n by means of a Bayesian estimator ϑ .

The qubits cannot distinguish temperatures that are much smaller or greater than E/k_B . We use the relative entropy between T_1, T_2 ,

$$D(T_1 \| T_2) = \sum_{n=0}^N P(n|T_1) \log_2 \frac{P(n|T_1)}{P(n|T_2)},$$

to define the temperatures that are indistinguishable from $T = 0$ and $T = \infty$.

REFERENCES

1. J. Rubio *et al.*, 2021, arXiv:2011.13018.
2. G. O. Alves and G. T. Landi, 2021, arXiv:2106.12072.
3. M. R. Jørgensen *et al.*, 2021, arXiv:2108.05901.
4. M. Mehboudi *et al.*, 2021, arXiv:2108.05932.

BAYESIAN ESTIMATION

The number of excitations measured, n , is used to calculate a posterior distribution

$$\begin{aligned} P(T|n) &= \frac{P(n|T)P^{(0)}(T)}{P(n)}, \\ P(n) &= \int_0^\infty dT P(n|T)P^{(0)}(T). \end{aligned}$$

We are then faced with three choices:

1. **Prior:** Most uninformative is Jeffreys' prior derived from the Fisher information $I(T)$

$$P^{(0)}(T) \propto \sqrt{I(T)} = \sqrt{\sum_n P(n|T) [\partial_T \ln P(n|T)]^2}.$$

2. **Estimators:** Bayesian estimators obtained by minimising an average cost function over all outcomes and temperatures.

| key | cost | estimator |
|------|-------------------------|---|
| (1) | $ \theta_n - T $ | median of $P(T n)$ |
| (1r) | $ \theta_n/T - 1 $ | median of $P(T n)/T$ |
| (2) | $(\theta_n - T)^2$ | $\langle T \rangle$ |
| (2r) | $(\theta_n/T - 1)^2$ | $\langle T^{-1} \rangle / \langle T^{-2} \rangle$ |
| (2l) | $\ln^2(\theta_n/T)$ | $\frac{E}{k_B} \exp \left[\int dT P(T n) \ln \left(\frac{k_B T}{E} \right) \right]$ |
| (md) | $-\delta(\theta_n - T)$ | $\arg \max_\theta P(\theta n)$ |

- 3 **Error measures:** For the sake of comparison all errors are converted to units of temperature and scaled to match the 90% confidence interval.

- Error from cost functions

$$\begin{aligned} \varepsilon_n &= f(\langle c(\vartheta_n, T) \rangle) \\ \bar{\varepsilon}(T) &= \sum_n P(n|T) \varepsilon_n \end{aligned}$$

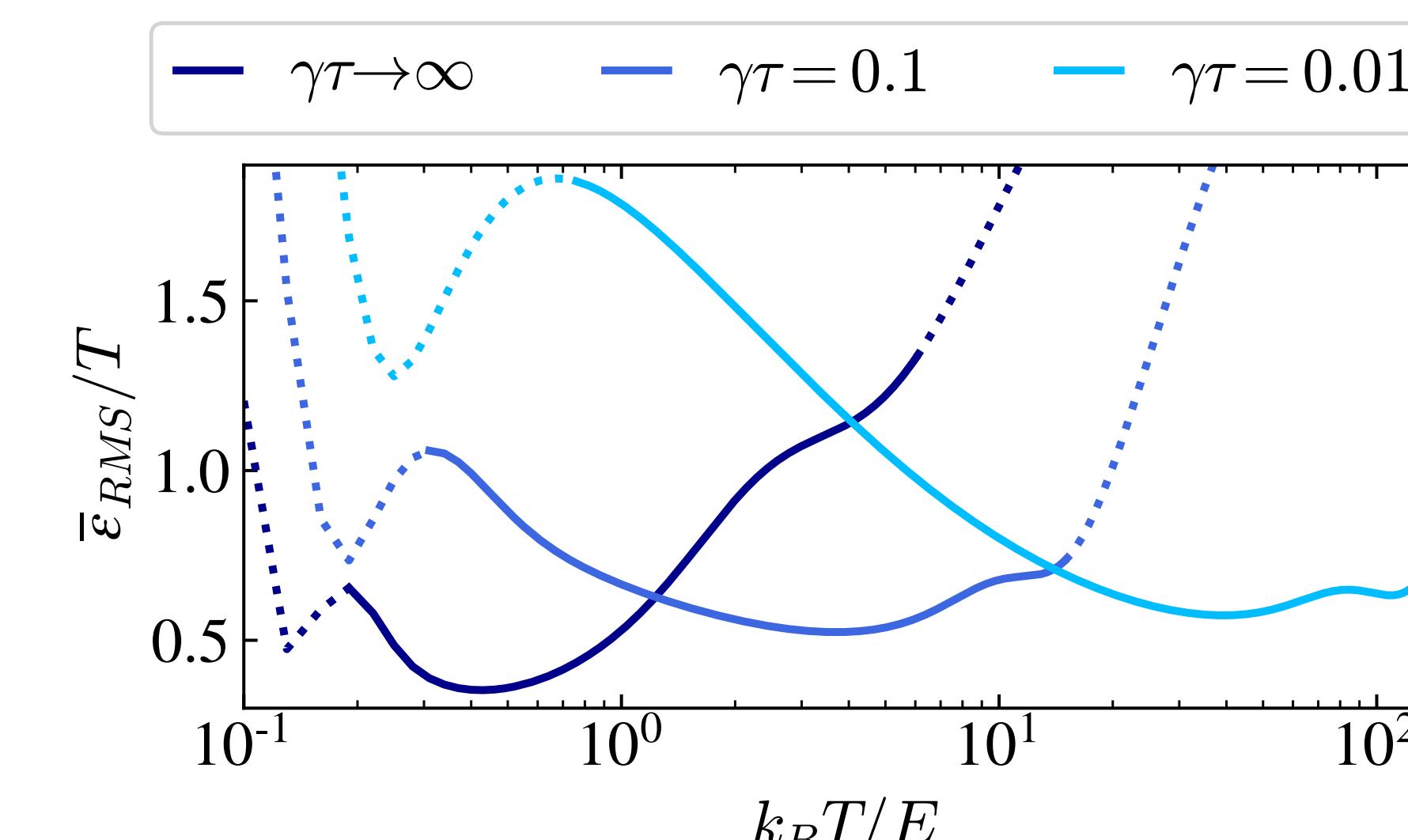
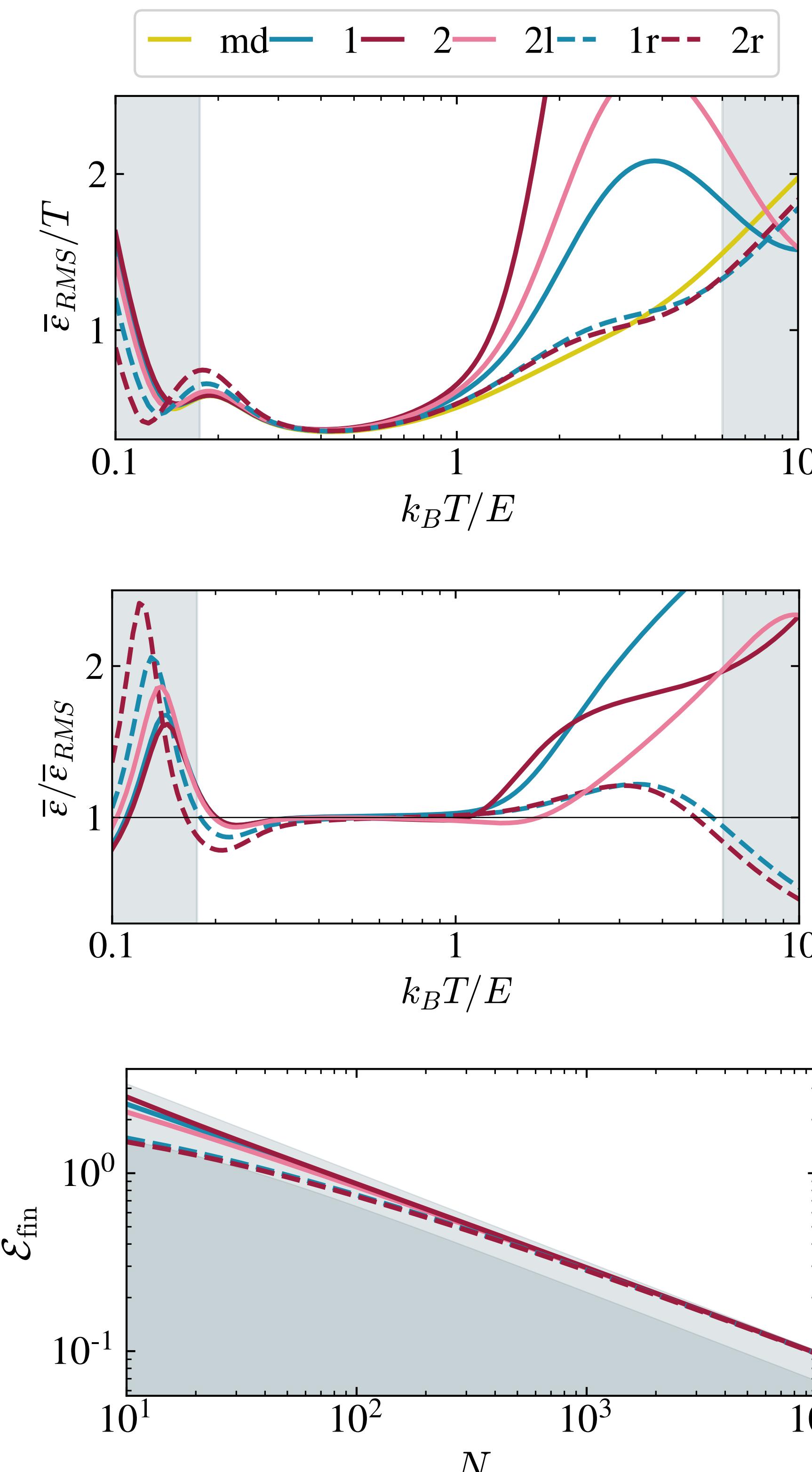
- Error based only on Posterior distribution

$$\int_0^{\theta_n^{90\%}} dT P(T|n) = 90\%$$

- Deviation from the true temperature

$$\bar{\varepsilon}_{\text{RMS}}(T) = 3.29 \sqrt{\sum_n P(n|T) (\vartheta_n - T)^2}$$

RESULTS



FUTURE RESEARCH

- **Adaptive schemes** with machine learning using Bayesian estimation
- Optimal protocols for **high temperature thermometry**

Estimators and Errors

- Shaded bars delimit the detectable temperature range
- Deviation from true temperature is not useful in a real experiment
- Relative estimators make the smallest errors at high temperatures and the error measures match the true error the best

Global Error Scaling

Take the average of the relative temperature error measures over all outcomes

$$\mathcal{E}_{\text{fin}}(\vartheta) = \sum_{n=0}^N P(n) \frac{\varepsilon_n}{\vartheta_n}.$$

The relative Cramér-Rao bound scaled to 90% confidence is then

$$\mathcal{E}_{\text{CRB}} = \int_{T_1}^{T_2} dT \frac{P^{(0)}(T) \bar{\varepsilon}_{\text{CRB}}(T)}{\mathcal{M} T} = \frac{6.58 \ln(T_2/T_1)}{\pi \sqrt{N} \mathcal{M}}.$$

Problem: This only works for the asymptotic limit.

Propose a global error benchmark on the relative temperature error based on van Trees bound

$$\mathcal{E}_{\text{rRMS}} \geq \frac{3.29}{\sqrt{(N+1)\pi^2/8 - N + 1}} \approx \frac{6.81}{\sqrt{N + 9.56}}.$$

Nonequilibrium Thermometry

Shorter τ extends and shifts the detectable range to higher temperatures. This is at the expense of the lowest achievable error

TAKEAWAY

