Lecture outline:

Introduction:

↳ spoken about limits of state transformation & work extraction [resource theories], now we consider the machines required for these transformations.

↳ Machine: state [Hamiltonians] of involved systems from the bath

+

Allowed operations on them.

[+repetitions] complexity levels

size
- ↳ dimension
- ↳ no. of systems

spectrum
- ↳ largest energy gap
- ↳ no. of distinct ΔE's
- ↳ spread of ΔE's
  - ↳ open question?!

1. any unitary

   ⋮

2. energy-pres with ideal work source

   ⋮

3. energ-pres with hot bath as source

   ⋮

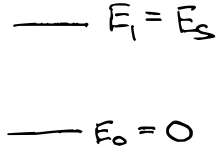
4. autonomous machines.

TAKEAWAY:

For some things [bounds to cooling, auto-machines at Carnot]: minimal complexity.
For others [saturating 2^{nd} law, cooling to $T \to 0^+$] ∞.

Working Example : Cooling a qubit.

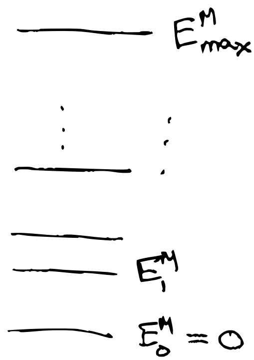
$H_S \equiv$



start at $\beta \rightarrow \rho_S = \dfrac{e^{-\beta H_S}}{Z}$

[can start elsewhere, analysis similar].

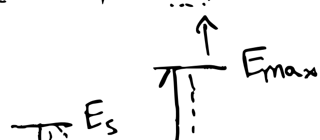
$H_M \equiv$



$$\mathrm{Tr}_M\left[U(\rho_S \otimes \tau_M)U^\dagger\right] = \rho_S' \quad \ldots \quad \boxed{\&\ \text{repeat}}$$

Q. How cold can we get?

A. Start with qubit M, swap states.



$$\begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \otimes \begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix} \Rightarrow \begin{pmatrix} pq & & & \\ & p(1-q) & & \\ & & (1-p)q & \\ & & & (1-p)(1-q) \end{pmatrix}$$

$\longrightarrow |01\rangle\langle 10| + |10\rangle\langle 01|$
$(+\, |00\rangle\langle 00| + |11\rangle\langle 11|)$

$$\begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix} \otimes \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \Leftarrow \begin{pmatrix} pq & & & \\ & (1-p)q & & \\ & & p(1-q) & \\ & & & (1-p)(1-q) \end{pmatrix}$$

This is the best possible with qubit machine

↳ exchanges Gibb's ratio of S & M.

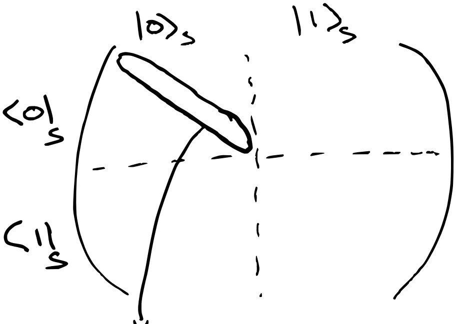
↳ $e^{-\beta_{final} E_S} = e^{-\beta E^M_{max}}$

$$\boxed{\beta_{final} = \beta \cdot \frac{E^M_{max}}{E_S}}$$

will turn out to be rather general.

↳ repetitions do not help.

↳ another view: permute the largest eigenvalues in ρ_{SM} to $|0\rangle\langle 0|_S$ subspace.



$\mathrm{Tr}\left[|0\rangle\langle 0|_S \cdot \mathrm{Tr}_M\left[\rho'_{SM}\right]\right]$

This sum is final ground pop of S

Q. What happens for larger machines?

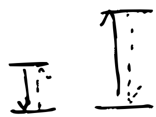
A. Exactly the permutation above.

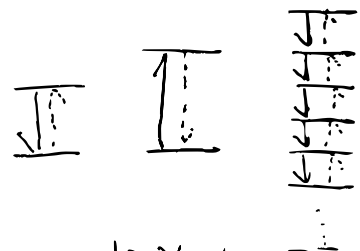
↳ but this mixes different swaps, of different Gibb's ratios $\Rightarrow$ β_{final} is less than $\beta \cdot \frac{E^M_{max}}{E_S}$.

$\boxed{\text{I:}}$ <u>To cool maximally, the simplest machine is best.</u>

Using ② , could we achieve the same?

↳ yes, because coherence is irrelevant here.





$$U_{SM} = |01\rangle\langle 10|_{SM} + |10\rangle\langle 01|_{SM} + \mathbb{1}^{rest}$$

$$U_{SMW} = |01\rangle\langle 10|_{SM} \otimes \Gamma_W^- + |10\rangle\langle 01| \otimes \Gamma_W^+ + \mathbb{1}_{rest}.$$

full swaps ⇒ only require incoherent resources.

Using ③ ?

↳ no, but the principle remains the same.

virtual qubit.

$$U_{SMH} = |010\rangle\langle 101|_{SMH} + |101\rangle\langle 010|_{SMH} + \mathbb{1}^{rest}_{SMH}$$

Gibbs ratio of virtual qubit:

$$\frac{p_{10}}{p_{01}} = \frac{p_1^M}{p_0^M} \cdot \frac{p_0^H}{p_1^H} = e^{-\beta E_M} e^{+\beta_H E_H} \quad \hookrightarrow E_M - E_S$$

$$e^{\beta_S E_S} \nearrow \Rightarrow \beta_S E_S \to \beta E_M - \beta_H (E_M - E_S) < \beta_M E_M$$

As $\beta_H \to \infty$, we get to the coherent limit.
$(T_H \to \infty)$

$\boxed{\text{II}}$ Coherent operations are better

1. The number of operations [time resource] increases for incoherent
2. However, the bound itself is attained as $\beta_H \to 0$.

key diff. between ② & ③ even for $\beta_H = 0$.
[single op of ② is asymptotic limit for ③, steadystate for ④]

③ & ④ are very similar, turn discrete rep's of single step into continuous interactions.

$$U_{SMH} \longrightarrow H^{int}_{SMH} = |010\rangle\langle 101| + |101\rangle\langle 010|$$

Thermalisation $\longrightarrow$ $C_t = (1-e^{-\gamma t})\tau + e^{-\gamma t}\rho$ ····· $\mathrm{Tr}_x[\rho]\otimes\tau$

$$\hookrightarrow \frac{d\rho}{dt} = \underbrace{\gamma_x(\tau-\rho)}_{}$$

$\hookrightarrow D_x(\rho)$, dissipator from bath x.

Conclusion:

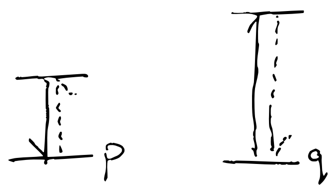
For just the question of cooling [or the attainability of any state change], the simplest machines suffice.

(!) For efficiency however .. that's another story.

Lecture 2:

Here we keep track of the work cost, and try to saturate Landauer/2nd law.

Back to qubit cooling:



$$\rho'_{SM} = U_{SWAP}\, \rho_{SM}\, U^{\dagger}$$

$$\langle E_i \rangle = (1-p) E_S + (1-q) E_B$$

$$\langle E_f \rangle = (1-p) E_B + (1-q) E_S$$

$$\therefore \Delta\langle E \rangle = W = (q-p)(E_B - E_S)$$

If S is thermal then $W \geq 0$ (same sign for terms above).

Now get S back to initial state [not the reverse operation]

$$\Rightarrow E_S = E'_B \Rightarrow W = 0!$$

Work has been lost $\Rightarrow$ irreversible.

Landauer erasure as an equality.

(S) (B) U_{SB} [arbitrary].

Initial $\rho_S \otimes \tau_B$ [no mutual info]

Final ρ'_{SB} [+ve mutual info].

$$W - \Delta F_S \begin{cases} \geq 0 \\ = I'(S;B) + S(\rho'_B \| \tau_B) \end{cases}$$

$$I'(S;B) = S(\rho'_S) + S(\rho'_B) - S(\rho'_{SB})$$

At heart of thermo irreversibility:

1. The loss of information to environment
   ↳ $I'(S;B)$
2. The finite deviation of machine from thermal $S(\rho'_B \| \tau_B)$, which is then lost when B equilibrates again. [max entropy, Jaynes].

In qubit example: B changes state. In fact, try cooling to $T \to 0 \Rightarrow q \sim 1-\varepsilon$

$\overline{}\ 1-p$ $\quad$ $\overline{}\ \varepsilon$

$\overline{}\ p$ $\quad$ $\uparrow E_B$ $\quad$ $\overline{}\ 1-\varepsilon$

$$D(\rho'_B \| \rho_B) = p \log \frac{p}{1-\varepsilon} + (1-p) \log \frac{(1-p)}{\varepsilon} \to \infty \text{ as } \varepsilon \to 0$$

$$W \sim (1-p)(E_B - E_S)$$

"An improved Landauer ..." ⇒ [Reeb & Wolf, 2014]

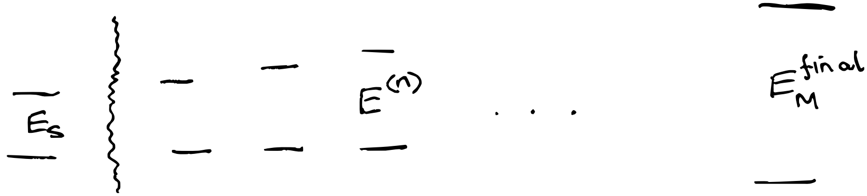
R&W: One cannot saturate Landauer with finite-dimensional ρ_B

Q1. How does error scale with d_B?

Q2. How does error depend on $\{\Delta E_B\}$

$\{E_B^i - E_B^j\}_{i,j}$.

Protocol for cooling closer to Landauer: split swaps into sequence. [Skrzypczyk et al. 2013]



N steps:

↳ key is to keep $\rho_B' \sim \rho_B \Rightarrow S(\rho_B' \| \rho_B)$ ~ small.

Easy choice (not optimal, Marti):

linear $E^{(n)} = E_S + n \cdot \left(\dfrac{E_M^{final} - E_S}{N} \right)$

↳ total no. of steps

$$W^{(n)} - \Delta F_S^{(n)} \sim O(N^{-2})$$

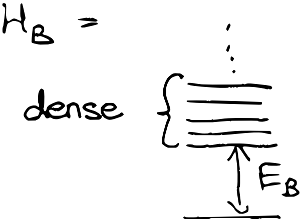
$$\Rightarrow W - \Delta F \sim O(1/N) \to 0 \quad \text{as} \quad N \to \infty.$$

$$\left(\frac{\beta^*/\beta - 1}{N} \right)$$

Properties of protocol:

1. $d_B \to \infty$ (taken together $U = U_1 \circ U_2 \circ \dots \circ U_N$

2. $\Delta E_B = E_B^{(n)} - E_B^{(n-1)} \to 0 \Rightarrow H_B =$



3. error $\sim \frac{1}{N}$
(uncorrelated)

R&W : error $\geq O\left(\frac{1}{\log^2 d}\right)$

$\log d \sim N \Rightarrow O\left(\frac{1}{N^2}\right)$

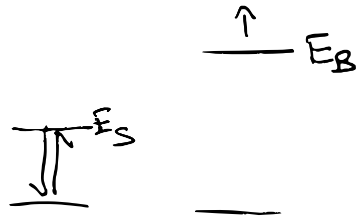
[can be correlated].

Importantly: all protocols demonstrate denseness of $\{E_B^i\}_i$, though this is not proven yet (to my knowledge).

Comments: incoherent extension $\Rightarrow$ add hot qubit to each step; repeat step a number of times.
[time increase].

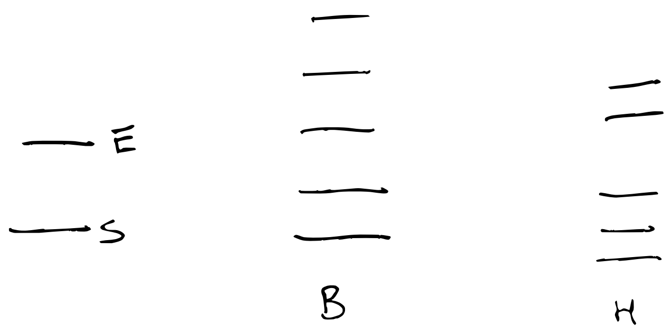
Final note [if time permits]

No-go for incoherent. Taranto et.al. 2021



(coherent)

With diverging energy : single swap → $T \sim 0^+$



$\rho_B = e^{-\beta H_B}$ $\qquad$ $\rho_H = e^{-\beta H_H}$

No single unitary (energy-pres) can go to $T = 0^+$

[reason : can only work in degenerate subspaces]

$$|100\rangle \leftrightarrow |0jk\rangle$$

$$E_S^1 = E_j^B + E_k^H$$

$$p_{0jk} = \frac{e^{-E_j^B \beta}}{Z_B} \cdot \frac{e^{-E_k^H \beta_H}}{Z_H}$$

REFERENCES:

1903.04970 (Clivaz et. al., PRL '19)

↳ Bounds on cooling for different control complexities

1306.4352 (Reeb & Wolf, NJP '14)

↳ Landauer principle with finite-size corrections (equality form of limit)

2106.05151 (Taranto et. al.)

↳ Trade-offs between complexity, time and energy for cooling to absolute zero