(please excuse my handwriting)

Wednesday, 18 August 2021 15:47

EQUILIBRATION AND THERMALIZATION

(some quick ideas)

e.g. 1704.06291 (Wilming et al)

① QUANTUM QUENCHES

- Initial pure state $|\phi\rangle$
  - Easy to prepare (product, ground state of H_0, ...)
- Hamiltonian suddenly switched to "H" (quench)
- Dynamics happen $e^{-itH}|\phi\rangle = |\phi(t)\rangle$
- After time t, observables can be measured: $\langle\phi(t)|A|\phi(t)\rangle = \langle A(t)\rangle$

Typically local observable

This is one of the most common settings in quantum many-body platforms.

Why?

- Often "relatively easy" to implement
- Show a variety of interesting physics → Thermalization (our focus), Localization, Transport, ...

---

② EQUILIBRATION AND THERMALIZATION

In "most" cases, something interesting happens to $\langle A(t)\rangle$

How long does it take?

1. EQ: It reaches a time-independent, steady value $\langle A(t)\rangle \to \langle A\rangle_{EQ}$
2. TH: That steady value corresponds to that of a Gibbs state

$$\langle A\rangle_{EQ} \simeq \mathrm{tr}\left[A\frac{e^{-\beta H}}{Z}\right] = \langle A\rangle_\beta$$

Temperature

Which temperature? Since energy is conserved, $\langle\phi|H|\phi\rangle = \mathrm{tr}\left[H\frac{e^{-\beta H}}{Z}\right]$

Exceptions
- Localized systems (Anderson/MBL)
- Exactly solvable models (free fermions, Bethe ansatz, ...)
- Quantum many-body scars
- ...

---

3 MECHANISM

Expand in energy eigenbasis

$$\nearrow H = \sum_j E_j |E_j\rangle\langle E_j|$$

$$\rightarrow |\phi\rangle = \sum_j c_j |E_j\rangle \qquad \text{(definitions)}$$

$$\searrow A = \sum_{j,k} A_{jk} |E_j\rangle\langle E_k|$$

$$\langle A(t)\rangle = \sum_{j,k} \underbrace{c_j c_k^*}_{\text{STATE}} \underbrace{A_{jk}}_{\text{OBS.}} e^{-it\underbrace{(E_j - E_k)}_{H}}$$

$$= \underbrace{\sum_{E_j = E_k} |c_j|^2 A_{jj}}_{\substack{\text{Thermalization} \\ \sim \langle A\rangle_\beta}} + \underbrace{\sum_{E_j \neq E_k} c_j c_k^* A_{jk} e^{-it(E_j - E_k)}}_{\substack{\text{Equilibration} \\ \sim 0}}$$

<u>Thermalization</u>: Eigenstate Thermalization Hypothesis (ETH) Srednicki '94

* For a fixed energy E, all $E_j \sim E$ s.t. (diagonal ETH)

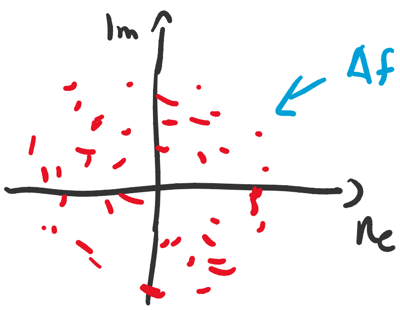
$$A_{jj} = \langle A\rangle_\beta + e^{-O(S(E))} \quad \leftarrow \text{Exp. small correction}$$

* Thus "Diagonal ensemble" $\sum_j |c_j|^2 A_{jj} \sim \langle A\rangle_\beta \sum_j |c_j^2| = \langle A\rangle_\beta \leftarrow$ Thermalization!

<u>Equilibration</u>: (more complicated)

$$\langle A(t)\rangle - \langle A\rangle_\beta = \sum_{E_j \neq E_k} c_j c_k^* A_{jk} e^{-it(E_j - E_k)}$$

* How to understand this? A huge number of complex numbers $c_j c_k^* A_{jk}$ oscillating at different frequencies $\omega \sim (E_j - E_k)$



After some time we have a chaotic but sort of "evenly distributed" cloud of complex numbers, that average to zero.

level

to zero.

* Evenly distributed? Why? $E_j - E_k \neq E_\ell - E_m$ level repulsion
  - Non-degenerate energy gaps, all with different values (Wigner-Dyson distribution)
  - Amplitudes: * $c_j = \langle E_j | \phi \rangle$ overlap of simple state with a very complex, highly entangled energy eigenstate $|E_j\rangle$, Very small.
    * Since $\sum_j |c_j|^2 = 1$, there are $e^{O(S)}$ many contributing c_j, all very small $e^{O(-S)}$.
  - Observable: obeys "off-diagonal ETH" Srednicki '94

$$A_{jk} = e^{-S/2} f(E, E_j - E_k) R_{jk}$$
$(j \neq k)$

small — some frequency dependence — Random

$e^{O(S)}$ numbers, all roughly randomly distributed, but with some structure

* What is the timescale? HARD QUESTION, likely no general answer

Suggestion: $t_{eq}^{-1} \propto \sqrt{\sum_{jk} |c_j|^2 |c_k|^2 A_{jk}^2}$ [Oliveira et al '16] NJP
(likely not always true)

④ LATE-TIME FLUCTUATIONS (a rigorous result)

[Reimann '08] [Short et al '08]

* Diagonal ensemble $\rho = \sum_j |c_j|^2 |E_j\rangle\langle E_j|$ (typically looks thermal)

* Difference w.r.t. $|\phi(t)\rangle$ $\langle A(t)\rangle - \mathrm{tr}[\rho A]$

This decreases with t, so we expect that as $t \to \infty$ its average value should vanish.

* We can upper bound this

$$\lim_{T\to\infty} \int_0^T \frac{1}{T} \left(\langle A(t)\rangle - \mathrm{tr}(\rho A) \right)^2 dt$$

$$= \sum_{j,k} \sum_{m,\ell} c_j c_k^* c_m c_\ell^* A_{jk} A_{\ell m} \lim_{T\to\infty} \int_0^T \frac{e^{-it[\underbrace{(E_j - E_k)}_{\text{Gaps}} - \underbrace{(E_\ell - E_m)}_{\text{Gaps}}]}}{T} dt$$

$$= \sum_{j,k}\sum_{m,l} c_j c_k^* c_m c_l^* A_{jk} A_{lm} \lim_{T\to\infty}\int_0^T \frac{e^{\cdots}}{T}\, dt$$

$$\nearrow \; = \delta_{jl}\,\delta_{km}$$

<u>Assumption</u>: non-degenerate energy gaps $(E_j - E_k) = (E_l - E_m) \iff j = l;\ k = m$

$$= \sum_{j,k}\sum_{m,l} c_j c_k^* c_m c_l^* A_{jk} A_{lm}\, \delta_{jl}\,\delta_{km}$$

$$= \sum_{j \neq m} |c_j|^2 |c_m|^2 |A_{jm}|^2$$

$$\leq \sum_{j,m} |c_j|^2 |c_m|^2 |A_{jm}|^2 = \mathrm{tr}\left[A \rho A^\dagger \rho\right]$$

$X = A\rho$
$Y = A^\dagger \rho$

Now use Cauchy-Schwarz ineq. $\mathrm{tr}\left[X Y^\dagger\right]^2 \leq \mathrm{tr}(X X^\dagger)\,\mathrm{tr}(Y Y^\dagger)$

ρ^2 positive

$$\leq \sqrt{\mathrm{tr}\left[A^\dagger A \rho^2\right]\mathrm{tr}\left[A A^\dagger \rho^2\right]} \leq \|A\|^2 \, \|\rho^2\|_1 = \|A\|^2\, \mathrm{tr}\left[\rho^2\right]$$

Hölder $\|PQ\|_1 \leq \|P\|\,\|Q\|_1$

What is $\mathrm{tr}[\rho^2]$? → Effective dimension
→ Inverse participation ratio
→ Purity of diagonal ensemble

Typically exp. small $\mathrm{tr}[\rho^2] \leq e^{-O(N)}$; N system size

Thus late-time fluctuations are VERY suppressed

$$\lim_{T\to\infty}\int \frac{dt}{T}\left(\langle A(t)\rangle - \mathrm{tr}[\rho A]\right)^2 \leq e^{-O(N)}$$

Q: How long does it take it to become exp. small?