Quantum Correlations in Time

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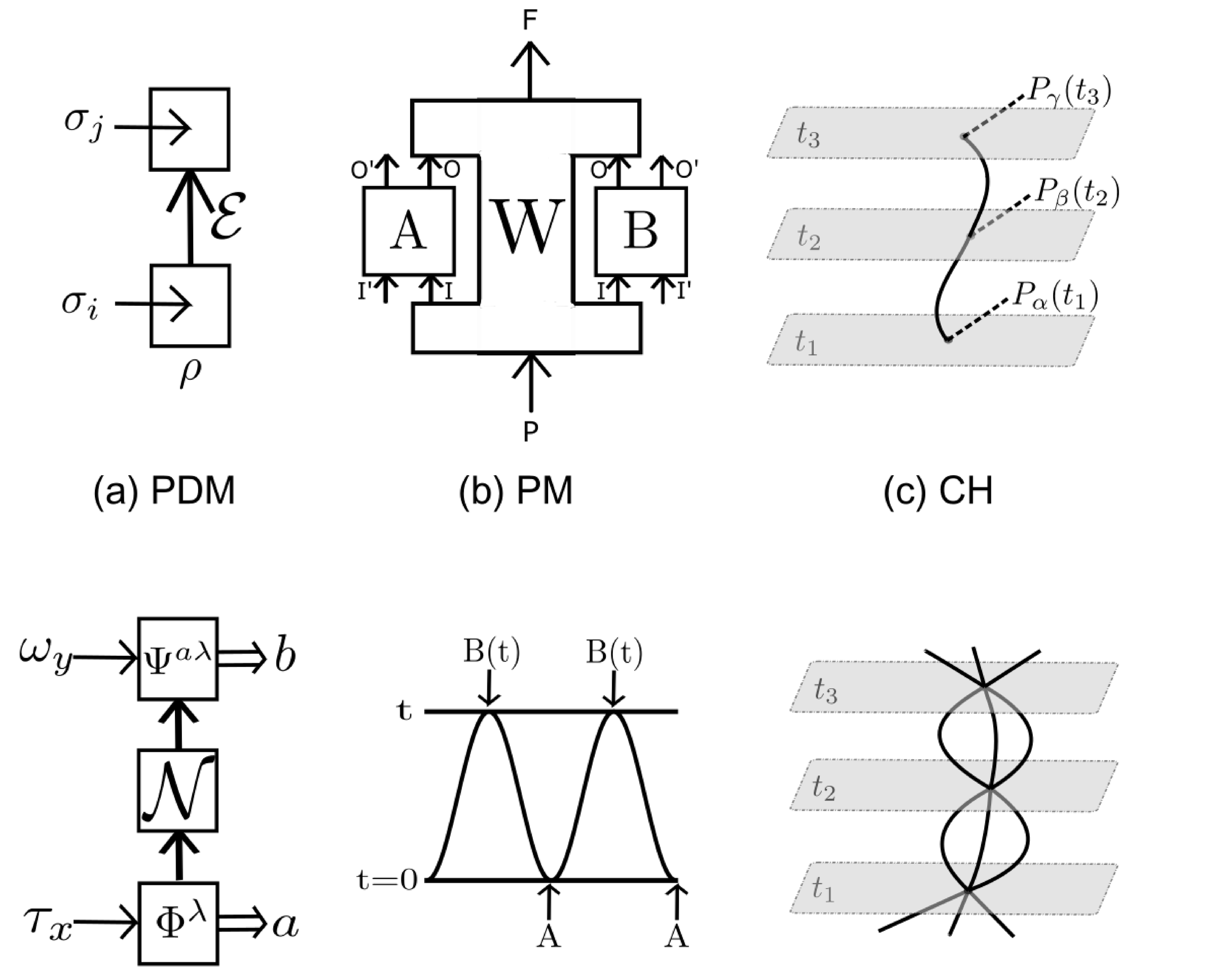
Summary

- We investigate quantum correlations in time in different approaches. We assume that temporal correlations should be treated in an even-handed manner with spatial correlations. We compare the pseudo-density matrix formalism with several other approaches: indefinite causal structures, consistent histories, generalised quantum games, out-of-time-order correlations(OTOCs), and path integrals. We establish close relationships among these space-time approaches in non-relativistic quantum theory, resulting in a unified picture. With the exception of amplitude-weighted correlations in the path integral formalism, in a given experiment, temporal correlations in the different approaches are operationally equivalent.

Figure Representation



(a) PDM (b) PM (c) CH

(d) QCSG (e) OTOC (f) PI

Pseudo-Density Matrix (PDM) Formulation

- A density matrix could be expressed as

$$\rho = \frac{1}{2^n}\sum_{i_1=0}^{3}\cdots\sum_{i_n=0}^{3}\langle\bigotimes_{j=1}^{n}\sigma_{i_j}\rangle\bigotimes_{j=1}^{n}\sigma_{i_j}. \tag{1}$$

- Consider a set of events $\{E_1,...,E_N\}$. At each event E_j, a measurement of a single qubit Pauli operator $\sigma_{i_j}\in\{\sigma_0,...,\sigma_3\}$ is made. For a particular choice of Pauli operators $\{\sigma_{i_j}\}_{j=1}^n$, $\langle\{\sigma_{i_j}\}_{j=1}^n\rangle$is defined as the expectation value of the product of the result of these measurements.
- The pseudo-density matrix is defined as

$$R = \frac{1}{2^n}\sum_{i_1=0}^{3}\cdots\sum_{i_n=0}^{3}\langle\{\sigma_{i_j}\}_{j=1}^n\rangle\bigotimes_{j=1}^{n}\sigma_{i_j}. \tag{2}$$

Process Matrix (PM): Indefinite Causal Structures

- Consider a global past P and a global future F. A process is defined as a linear transformation take two CPTP maps $\mathcal{A}: A_I\otimes A_I'\to A_O\otimes A_O'$ and $\mathcal{B}: B_I\otimes B_I'\to B_O\otimes B_O'$ to a CPTP map $\mathcal{G}_{\mathcal{A},\mathcal{B}}: A_I'\otimes B_I'\otimes P\to A_O'\otimes B_O'\otimes F$ without acting on A_I', A_O', B_I', B_O'. Specifically, it is a transformation that act on $P\otimes A_I\otimes A_O\otimes B_I\otimes B_O\otimes F$.
- We introduce the Choi-Jamiołkowski isomorphism to represent the process in the matrix formalism. Recall that for a completely positive map $\mathcal{M}^A: A_I\to A_O$, its corresponding Choi-Jamiołkowski matrix is given as $\mathfrak{C}(\mathcal{M})\equiv[\mathcal{I}\otimes\mathcal{M}^A(|1\rangle\langle 1|)]\in A_I\otimes A_O$ with $\mathcal{I}$ as the identity map and $|1\rangle\!\rangle = |1\rangle\!\rangle^{A_IA_I}\equiv\sum_j|j\rangle^{A_I}\otimes|j\rangle^{A_I}\in\mathcal{H}^{A_I}\otimes\mathcal{H}^{A_I}$ is the non-normalised maximally entangled state. The inverse is given as $\mathcal{M}(\rho^{A_I})=\mathrm{Tr}[(\rho^{A_I}\otimes 1^{A_O})M^{A_IA_O}]$ where 1^{A_O} is the identity matrix on $\mathcal{H}^{A_O}$.
- Then $A=\mathfrak{C}(\mathcal{A})$, $B=\mathfrak{C}(\mathcal{B})$, and $G_{A,B}=\mathfrak{C}(\mathcal{G}_{\mathcal{A},\mathcal{B}})$ are the corresponding CJ representations. We have

$$G_{A,B}=\mathrm{Tr}_{A_IA_OB_IB_O}[W^{T_{A_IA_OB_IB_O}}(A\otimes B)], \tag{3}$$

where the process matrix is defined as $W\in P\otimes A_I\otimes A_O\otimes B_I\otimes B_O\otimes F$, $T_{A_IA_OB_IB_O}$ is the partial transposition on the subsystems A_I, A_O, B_I, B_O, and we leave identity matrices on the rest subsystems implicit.
- **Correlation Analysis** Consider a single qubit ρ evolving under U. The correlations from the process matrix are given by

$$p(\Sigma_i^{A_IA_O},\Sigma_j^{B_IB_O})=\mathrm{Tr}[(\Sigma_i^{A_IA_O}\otimes\Sigma_j^{B_IB_O})W]=\frac{1}{2}\mathrm{Tr}[\sigma_jU\sigma_iU^\dagger], \tag{4}$$

while the correlations from the pseudo-density matrix are given as

$$\langle\{\sigma_i,\sigma_j\}\rangle=\frac{1}{2}\left(\mathrm{Tr}[\sigma_jU\sigma_i\rho U^\dagger]+\mathrm{Tr}[\sigma_jU\rho\sigma_iU^\dagger]\right)=\frac{1}{2}\mathrm{Tr}[\sigma_jU\sigma_iU^\dagger]. \tag{5}$$

Consistent Histories (CH)

- Suppose that the system is in the state ρ at the initial time t_0. Consider a set of histories $[\alpha]=[\alpha_1,\alpha_2,\cdots,\alpha_n]$ consisting of n projections $\{P^k_{\alpha_k}(t_k)\}_{k=1}^n$ at times $t_1<t_2<\cdots<t_n$.zzThen the decoherence functional is defined as

$$D([\alpha],[\alpha'])=\mathrm{Tr}[P^n_{\alpha_n}(t_n)\cdots P^1_{\alpha_1}(t_1)\rho P^1_{\alpha_1'}(t_1)\cdots P^n_{\alpha_n'}(t_n)], \tag{6}$$

where

$$P^k_{\alpha_k}(t_k)=e^{i(t_k-t_0)H}P^k_{\alpha_k}e^{-i(t_k-t_0)H}. \tag{7}$$

- Consider an n-qubit pseudo-density matrix as a single qubit evolving at n times. For each event, we make a single-qubit Pauli measurement σ_{i_k} at the time t_k. We can separate the measurement σ_{i_k} into two projection operators $P^{+1}_{i_k}=\frac{1}{2}(I+\sigma_{i_k})$ and $P^{-1}_{i_k}=\frac{1}{2}(I-\sigma_{i_k})$ with its outcomes ±1. A pseudo-density matrix is built upon measurement correlations $\langle\{\sigma_{i_k}\}_{k=1}^n\rangle$. Theses correlations can be given in terms of decoherence functionals as

$$\begin{aligned}\langle\{\sigma_{i_k}\}_{k=1}^n\rangle &= \sum_{\alpha_1,...,\alpha_n}\alpha_1\cdots\alpha_n\,\mathrm{Tr}[P^{\alpha_n}_{i_n}U_{n-1}\cdots U_1P^{\alpha_1}_{i_1}\rho P^{\alpha_1}_{i_1}U_1^\dagger\cdots U_{n-1}^\dagger P^{\alpha_n}_{i_n}]\\ &=\sum_{\alpha_1,...,\alpha_n}\alpha_1\cdots\alpha_n p(\alpha_1,\dots,\alpha_n)=\sum_{\alpha_1,...,\alpha_n}\alpha_1\cdots\alpha_n D([\alpha],[\alpha]),\end{aligned} \tag{8}$$

where $D([\alpha],[\alpha])$ is the diagonal terms of decoherence functional with $[\alpha]=[\alpha_1,\dots,\alpha_n]$.

Quantum-Classical Signalling Game (QCSG)

- Instead of two players Alice and Bob, we consider only one player Abby at two successive instants in time for quantum-classical signalling games [?] as

$$\overrightarrow{qcsg}=\langle\{\tau^x\},\{\omega^y\};\mathcal{A},\mathcal{B};l\rangle. \tag{9}$$

For admissible quantum strategies, suppose Abby at t_1 receives τ^x_X and makes a measurement of instruments $\{\Phi^{a|\lambda}_{X\to A}\}$, and gains the outcome a. The quantum output goes through the quantum memory $\mathcal{N}: A\to B$. The output of the memory and ω^y_Y received by Abby at t_2 are fed into a measurement $\{\Psi^{b|a,\lambda}_{BY}\}$, with outcome b. Then

$$p_q(a,b|x,y)=\sum_\lambda\pi(\lambda)\,\mathrm{Tr}[(\{(\mathcal{N}_{A\to B}\circ\Phi^{a|\lambda}_{X\to A})(\tau^x_X)\}\otimes\omega^y_Y)\Psi^{b|a,\lambda}_{BY}]. \tag{10}$$

- Assume ω^y_Y to be trivial. For Abby at the initial time and the later time, we consider $\Phi^a_{X\to A}: \tau^x_X\to\sum_i M^a_i\tau^x_XM^{a\dagger}_i$, $\sum M^{a\dagger}_iM^a_i=1_{\mathcal{H}^A}$. Between two times, the transformation from A to B is given by $\mathcal{N}:\rho_A\to\sum_jN_j\rho_AN_j^\dagger$ with $\sum_jN_j^\dagger N_j=1_{\mathcal{H}^A}$. Then

$$p_q(a,b|x,y)=\mathrm{Tr}[\{(\mathcal{N}_{A\to B}\circ\Phi^a_{X\to A})(\tau^x_X)\}\Psi^{b|a}_B]=\sum_{ijk}\mathrm{Tr}[N_jM^a_i\tau^x_XM^{a\dagger}_iN^\dagger_j\Psi^{b|a}_B]. \tag{11}$$

Out-of-Time-Order Correlation (OTOC)

- Consider local operators W and V. With a Hamiltonian H of the system, the Heisenberg representation of the operator W is given as $W(t)=e^{iHt}We^{-iHt}$. Out-of-time-order correlation functions (OTOCs) are usually defined as

$$\langle VW(t)V^\dagger W^\dagger(t)\rangle=\langle VU(t)^\dagger WU(t)V^\dagger U^\dagger(t)W^\dagger U(t)], \tag{12}$$

where $U(t)=e^{-iHt}$ is the unitary evolution operator and the correlation is evaluated on the thermal state $\langle\cdot\rangle=\mathrm{Tr}[e^{-\beta H}\cdot]/\,\mathrm{Tr}[e^{-\beta H}]$.
- Consider a qubit evolving in time and backward. In particular, we measure A at t_1, B at t_2 and A again at t_3 and assume the evolution forwards is described by U and backward $U^\dagger$. Then the probability is given by

$$\mathrm{Tr}[AU^\dagger BUA\rho A^\dagger U^\dagger B^\dagger UA^\dagger]=\mathrm{Tr}[AB(t)A\rho A^\dagger B^\dagger(t)A^\dagger]. \tag{13}$$

If we assume that $AA^\dagger=A$, $\rho=\frac{1}{d}$, Eqn. (13) will reduce to the OTOC.

Path Integral (PI)

- Two-point correlations functions in the path integral formalism is defined as

$$\langle q(t_1)q(t_2)\rangle=\frac{\int[\mathrm{d}q(t)]q(t_1)q(t_2)\exp[-\mathcal{S}(\boldsymbol{q})/\hbar]}{\int[\mathrm{d}q(t)]\exp[-\mathcal{S}(\boldsymbol{q})/\hbar]}. \tag{14}$$

- In the Gaussian representation of pseudo-density matrices, temporal correlation for q_1 at t_1 and q_2 at t_2 with the evolution U and the initial state $|q_1\rangle$ is given as

$$\langle\{q_1,q_2\}\rangle=\int\mathrm{d}q_1\mathrm{d}q_2q_1q_2\left|\int_{q(t_1)=q_1}^{q(t_2)=q_2}[\mathrm{d}q(t)]\exp[-\mathcal{S}(\boldsymbol{q})/\hbar]\right|^2. \tag{15}$$