Computable Rényi Mutual Information: Area Laws and Correlations

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Abstract

The **mutual information** is a **measure of classical and quantum correlations** in mixed bipartite quantum states. It is also relevant in quantum many-body physics, by virtue of satisfying an area law for thermal states and bounding all correlation functions. However, calculating it exactly or approximately is often challenging in practice. Here, we consider alternative definitions based on Rényi divergences [2], which can be expressed as **variational** problems efficiently solvable for families of states like **matrix product operators**. These definitions still have an information theoretic meaning. In particular, we show that they obey a **thermal area law** in great generality, and that they upper **bound all correlation functions**. We also investigate their behavior on certain tensor network states and on classical thermal distributions [3].

Rényi Divergences

Generalizations of relative entropy, e.g.,

- Geometric

$$\widehat{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \operatorname{Tr}\left[\sigma\left(\sigma^{-1/2}\rho\sigma^{-1/2}\right)^\alpha\right] \tag{1}$$

- Maximal

$$\begin{aligned}\widehat{D}_\alpha(\rho\|\sigma) \to D_\infty(\rho\|\sigma) &= \log\inf\{\lambda : \rho \leq \lambda\sigma\} &(2)\\ &= \log\inf\{\lambda : 0 \leq \langle\psi|\,\lambda\sigma-\rho\,|\psi\rangle\,\forall\psi\} &(3)\\ &= \log\inf\{\lambda : 0 \leq \inf_{|\psi\rangle}\langle\psi|\,\lambda\sigma-\rho\,|\psi\rangle\} &(4)\end{aligned}$$

- Measured [1]

$$\begin{aligned}D_\alpha^{\mathbb{M}}(\rho\|\sigma) &= \sup_{(\chi,M)}\left(P_{\rho,M}\|P_{\sigma,M}\right) &(5)\\ &= \frac{1}{\alpha-1}\log\sup_{\omega>0}\alpha\operatorname{Tr}\left[\rho\omega^{\alpha-1}\right]+(1-\alpha)\operatorname{Tr}[\sigma\omega^\alpha] &(6)\end{aligned}$$

Advantage: Variational expression for measured and maximal Rényi divergence

⇒ **Computable** with tensor network techniques

Correlation Functions

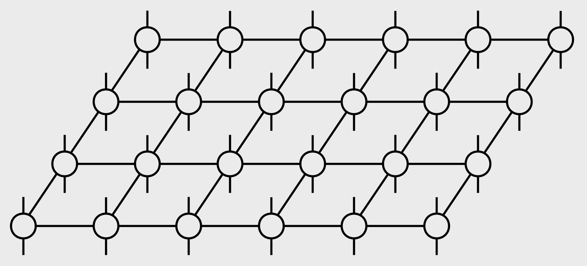
Mutual information bounds correlation functions [4]

$$\mathcal{C}(M_A, M_B) = \langle M_A \otimes M_B\rangle - \langle M_A\rangle\langle M_B\rangle \tag{7}$$

Rényi generalization:

$$I_\alpha^{\mathbb{M}}(A:B) \geq \frac{\min\{1,\alpha\}\mathcal{C}(M_A,M_B)^2}{2\|M_A\|^2\|M_B\|^2} \tag{8}$$

Area Law in PEPDOs



- PEPDOS are inspired by slightly correlated/entangled states
- **Question:** Area law for Rényi mutual information?
- **Answer:** Yes
  - assuming a purification
  - only for $I_2^{\mathbb{M}}(A:B)$

  Then:

$$I_\alpha^{\mathbb{M}}(A:B) \leq 2|\partial A|\log D \tag{9}$$

Conclusion

We propose a new definition of Rényi mutual information to overcome the computational difficulties of the von Neumann mutual information and give techniques to calculate these quantities for states given as matrix product operators. For these quantities we check whether they have similar information theoretic properties as their von Neumann counterpart. We prove thermal area laws in great generality, show how they behave in certain tensor networks states and that they bound correlation functions.

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Mutual Information

Von Neumann mutual information:

$$\begin{aligned}I(A:B) &= S(\rho_A)+S(\rho_B)-S(\rho_{AB}) &(10)\\ &= D(\rho_{AB}\|\rho_A\otimes\rho_B) &(11)\end{aligned}$$

Here, $S(\rho) = -\operatorname{Tr}[\rho\log\rho]$ and $D(\rho\|\sigma) = \operatorname{Tr}[\rho(\log\rho-\log\sigma)]$

Rényi generalization:

$$\boxed{\widehat{I}_\alpha(A:B) := \widehat{D}_\alpha(\rho_{AB}\|\rho_A\otimes\rho_B)} \tag{12}$$

or analogously with other Rényi divergences.

Basic properties:

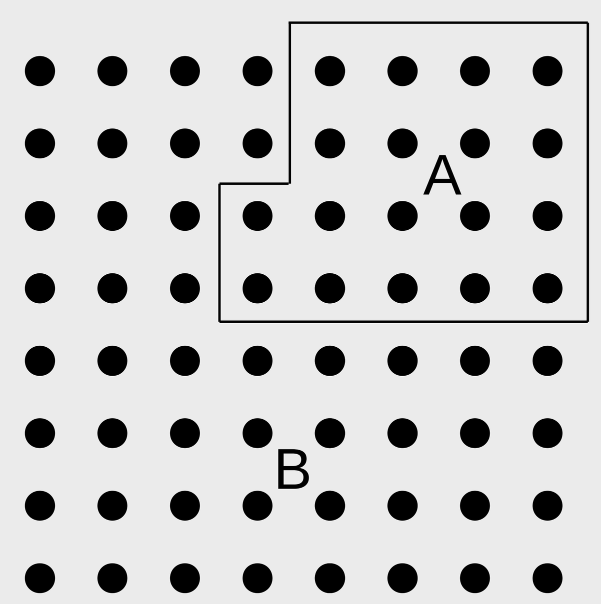
- Nonnegativity

$$\widehat{I}_\alpha(A:B) \geq 0 \tag{13}$$

- Data-processing inequality for $\mathcal{E} = \mathcal{N}\otimes\mathcal{M}$

$$\widehat{I}_\alpha(A:B)_\rho \geq \widehat{I}_\alpha(A:B)_{\mathcal{E}(\rho)} = \widehat{D}_\alpha(\mathcal{N}\otimes\mathcal{M}(\rho_{AB})\|\mathcal{N}(\rho_A)\otimes\mathcal{M}(\rho_B)) \tag{14}$$

Thermal Area Laws



- Local Hamiltonian
- Bulk terms H_A, H_B supported on A, B
- Interaction term H_I, together:

$$H = H_A + H_B + H_I \tag{15}$$

- **Intuition:** Interaction only on boundary ⇒ Mutual information scales with boundary [4]

$$I(A:B) \leq 2\beta\|H_I\| \tag{16}$$

- Rényi generalization in several cases
  - One dimension
  - High temperature
  - Commuting Hamiltonians
  - Classical states (temperature independent bound)

References

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[3] Samuel O. Scalet et al. *Computable Rényi mutual information: Area laws and correlations.* 2021. arXiv: `2103.01709 [quant-ph]`.

[4] Michael M. Wolf et al. "Area Laws in Quantum Systems: Mutual Information and Correlations". In: *Phys. Rev. Lett.* 100 (7 Feb. 2008), p. 070502.