Stochastic Path Integral Analysis of the Continuously Monitored Simple Harmonic Oscillator

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Abstract and outline

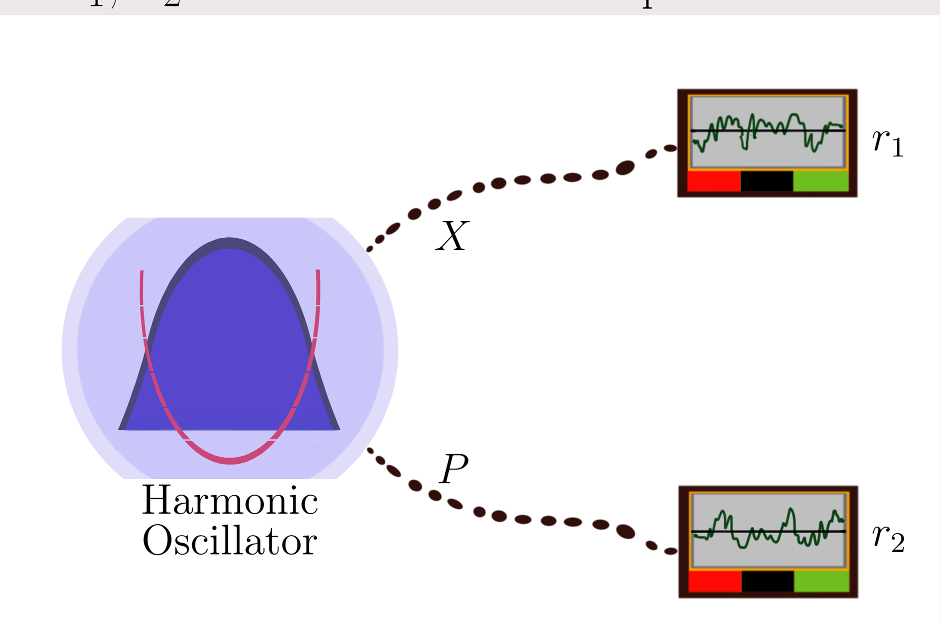
We consider the evolution of a harmonic oscillator in general Gaussian states undergoing simultaneous weak continuous position and momentum measurements. The expectation values of position and momentum show stochastic evolution while the covariance matrix elements show deterministic evolution and converge to their steady states. We apply the Chantasri-Dressel-Jordan stochastic path integral formalism to express the probability density of a sequence of readouts as a path integral of the exponential of a stochastic action. Extremization of the action gives us the most-likely paths. We find the analytical solutions for these most-likely paths for the steady state values of the covariance matrix elements. We also analyze the energetics of the measurement process and characterize final state probability densities starting from an initial state. We confirm our results using simulations.

System description

- System Hamiltonian is $\hat{H} = \frac{1}{2}(\hat{X}^2 + \hat{P}^2)$.
- Weak continuous measurements (in $d\tau$ intervals) of Gaussian type with Kraus operators

$$\hat{M}_X(r_1) = \left(\frac{d\tau}{2\pi\mathcal{T}_1}\right)^{\frac{1}{4}} \exp\left[-\frac{d\tau}{4\mathcal{T}_1}(r_1\mathbb{1} - \hat{X})^2\right] \quad \hat{M}_P(r_2) = \left(\frac{d\tau}{2\pi\mathcal{T}_2}\right)^{\frac{1}{4}} \exp\left[-\frac{d\tau}{4\mathcal{T}_2}(r_2\mathbb{1} - \hat{P})^2\right].$$

r_1, r_2 are the readouts of position and momentum measurements.

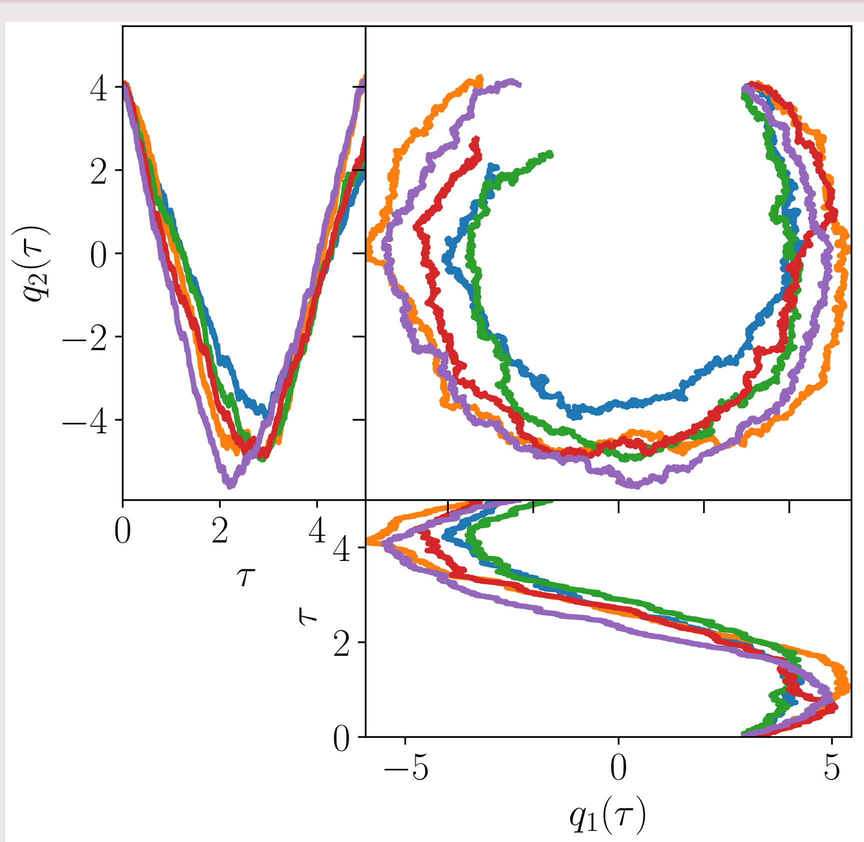


- $\mathcal{T}_1$ and $\mathcal{T}_2$ ($\gg d\tau$) are collapse time scales of the X and P measurements.
- The unitary dynamics and the Kraus operators preserve the Gaussianity of the state.
- We define $(\langle\hat{X}\rangle, \langle\hat{P}\rangle)^\top = (q_1, q_2)^\top = \boldsymbol{q}$.
- Covariance matrix $\Gamma = \begin{pmatrix} 2\mathrm{Var}(\hat{X}) & 2\mathrm{Cov}(\hat{X},\hat{P}) \\ \langle\hat{X}\hat{P}+\hat{P}\hat{X}\rangle - 2\langle\hat{X}\rangle\langle\hat{P}\rangle & 2\mathrm{Var}(\hat{P}) \end{pmatrix}$.

For pure states $\det\Gamma = 1$.

State evolution

- Sample trajectories from the same initial state are shown for $\mathcal{T}_1 = \mathcal{T}_2 = 1$ and $\tau_f = 5.00$.
- In this case, Γ converges to the $\mathbb{1}_2$ at $\tau \to \infty$. For simplicity we assume $\Gamma = \mathbb{1}_2$.
- The quadratures ($\boldsymbol{q}$) show stochasticity due to measurement back action.
- Spiral trajectories in phase space signify energy gain/loss due to measurements.



Chantasri-Dressel-Jordan stochastic path integral formalism

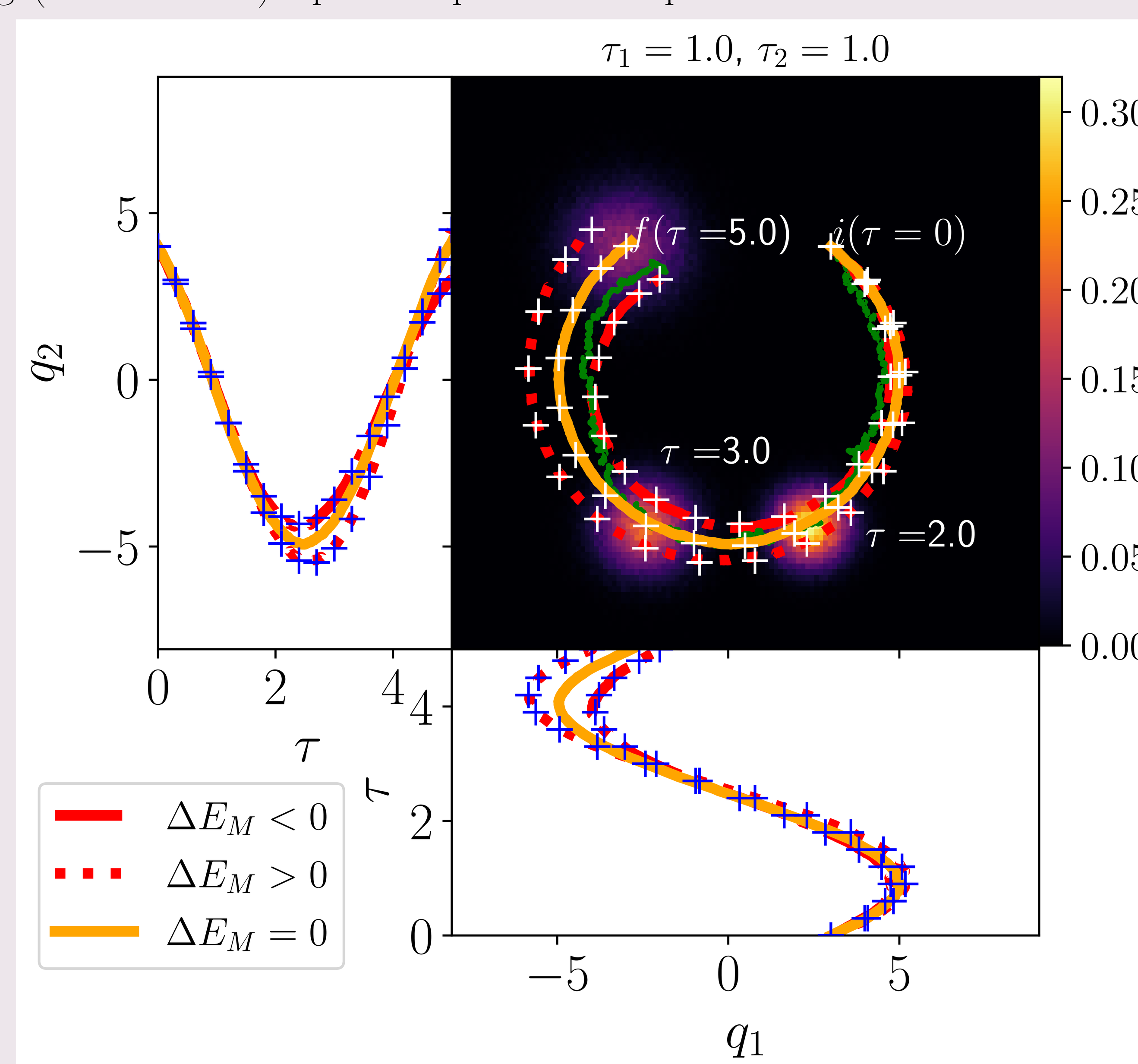
- The CDJ formalism lets us calculate the probability density associated with a sequence of readouts $\{\boldsymbol{r}_k\}$.
- $\boldsymbol{q}(0) = \boldsymbol{q}_i \to \boldsymbol{q}(\tau_f) = \boldsymbol{q}_f$ through intermediate states $\{\boldsymbol{q}_k\}$ and readouts $\{\boldsymbol{r}\}$.
- The CDJ formalism tells us

$$\mathcal{P}(\{\boldsymbol{q}_k\},\{\boldsymbol{r}_k\}) = \int \mathcal{D}\boldsymbol{p}\, e^{\mathcal{S}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}]} = \int \mathcal{D}\boldsymbol{p}\, \exp\left[\int_0^{\tau_f} d\tau(-\boldsymbol{p}\cdot\dot{\boldsymbol{q}} + \mathcal{H}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}))\right].$$

- $\boldsymbol{p}$ is momentum conjugate to $\boldsymbol{q}$, not to be confused with the mechanical momentum.
- $\mathcal{S}$ and $\mathcal{H}$ are the stochastic action and stochastic Hamiltonian respectively.
- $\delta\mathcal{S} = 0$ gives the most-likely readouts. The corresponding trajectory is the most-likely path or the optimal path.
- For the steady state of the covariance matrix elements, the most-likely paths can be solved for analytically.

Optimal paths

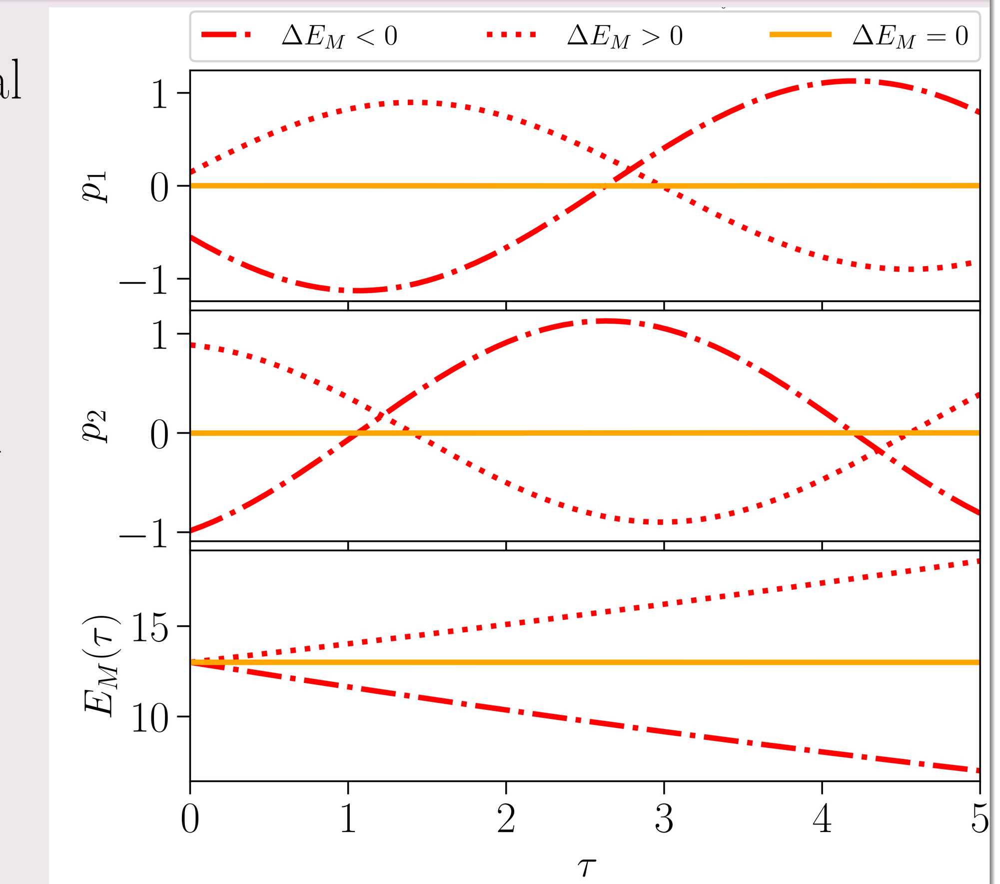
- The energy conserving optimal path (yellow solid) is circular in phase space, as the distance from the origin corresponds to the expectation value of the mechanical energy. The energy decreasing (red dash-dotted) and energy increasing (red dashed) optimal paths are spirals.



- A sample stochastic trajectory (green) is also shown.
- Averages of simulated clustered trajectories (denoted as '+') for the same boundary conditions show very good agreement with the analytical results.
- Histograms show the diffusion of trajectories at three different times. The colorbar denotes the probability density of trajectories.
- The energy conserving trajectory is the the globally most-likely path.

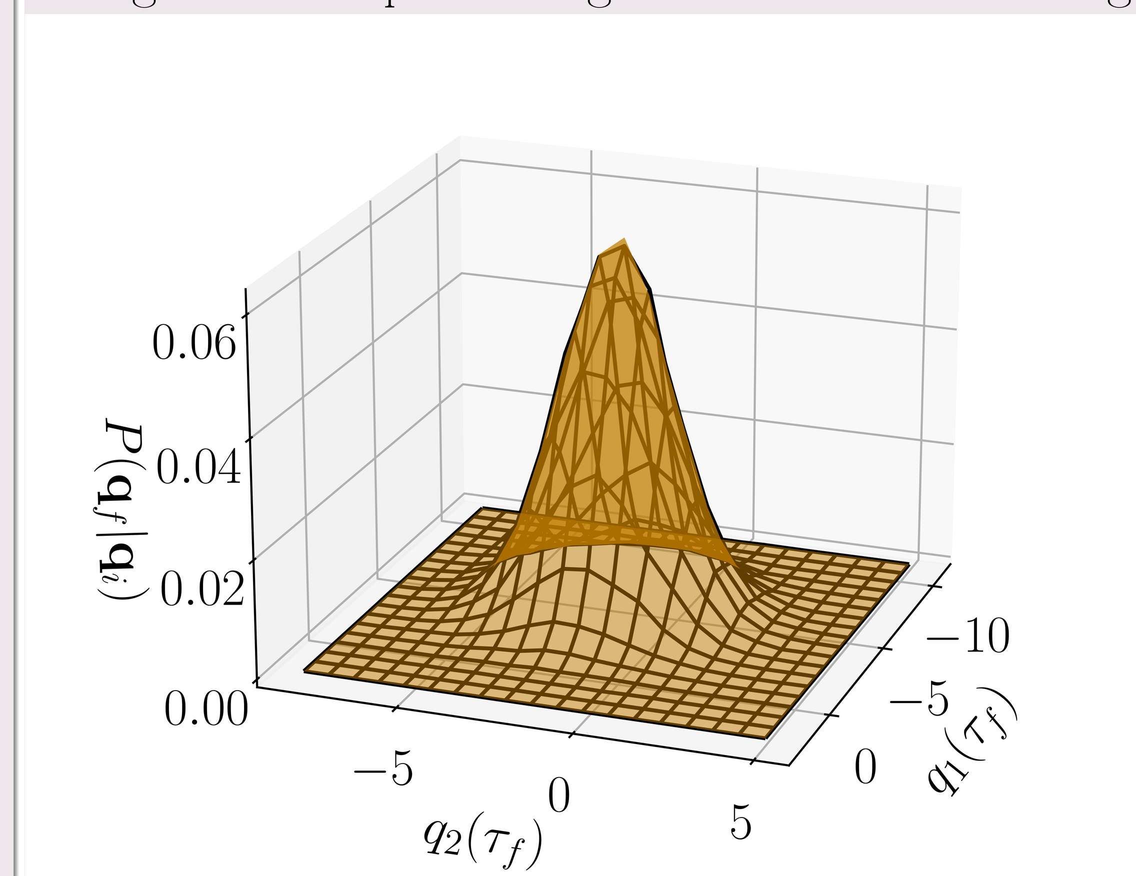
Energetics

- The conjugate momenta are sinusoidal while the energy conserving path corresponds to $\boldsymbol{p} = 0$.
- The expectation value of the mechanical energy is $E_M(\tau) = \frac{1}{2} + \frac{1}{2}\Big\{\Big(\mathcal{Q}_{1f} + \frac{\tau}{\tau_f}(q_{1f} - \mathcal{Q}_{1f})\Big)^2 + \Big(\mathcal{Q}_{2f} + \frac{\tau}{\tau_f}(q_{2f} - \mathcal{Q}_{2f})\Big)^2\Big\}$.
- $\mathcal{Q}_{1f} = q_{1i}\cos\tau_f + q_{2i}\sin\tau_f$ and $\mathcal{Q}_{2f} = -q_{1i}\sin\tau_f + q_{2i}\cos\tau_f$ denote the final state along the energy conserving optimal path.



Final state probability densities

Using the CDJ path integral formalism we also get $P(\boldsymbol{q}_f|\boldsymbol{q}_i)$.



- For $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$ and $\Gamma = \mathbb{1}_2$,

$$P(\boldsymbol{q}_f|\boldsymbol{q}_i) = \frac{2\mathcal{T}}{\pi\tau_f}\exp\left\{-\frac{2\mathcal{T}}{\tau_f}(q_{1f} - \mathcal{Q}_{1f})^2\right\} \times \exp\left\{-\frac{2\mathcal{T}}{\tau_f}(q_{2f} - \mathcal{Q}_{2f})^2\right\}$$

- Analytical results (orange surface) show good agreement with simulated results (black wireframe).

Summary and discussions

- Using Chantasri-Dressel-Jordan stochastic path integral formalism, first time for a continuous variable system, we completely characterize the measurement statistics of a continuously monitored general Gaussian state harmonic oscillator.
- Our optimal path description and analysis of the energetics can be useful for e.g. feedback control and cooling of resonators.
- Our work also provides a new way to connect optomechanical elements with novel quantum technologies like quantum measurement engines and refrigerators.

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