Thermodynamics in Superconducting Circuits

I. Superconducting Circuits [1-3]

$\boxed{\hbar = 1}$

Substance that conducts electricity without resistance

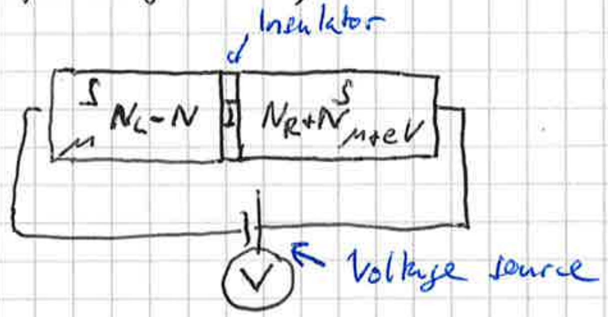
low energy excitations can be engineered!

Aluminium
$T_c \sim 1.2\,K$, $2\Delta \sim 82\,GHz$,
$\omega_p \sim 3.6 \cdot 10^6\,GHz$
Niobium
$T_c \sim 9.2K$, $2\Delta \sim 737\,GHz$
$\omega_p \sim 2.2 \cdot 10^6\,GHz$

- Ground state: Electrons paired in Cooper pairs (CP) with $E = 2\mu$
- Excitations in bulk superconductors:
  - Breaking a Cooper pair: 2Δ ← Superconducting gap
  - Density fluctuations: ω_p ← Plasma frequency

Engineering excitations

i) Coupling two superconductors (Josephson junction) [1]



$\hat{N}$: number of CPs that tunneled from left to right

$$\hat{H}_J = -\frac{E_J}{2} \sum_{N=-\infty}^{\infty} \left(|N\rangle\langle N+1| + |N+1\rangle\langle N| \right)$$

(∞ ← large superconductors)

$$= -\frac{E_J}{2}\left(e^{-i\hat{\theta}} + e^{i\hat{\theta}}\right) = -E_J \cos(\hat{\theta})$$

- Phase-operator

$$e^{i\hat{\theta}} = \sum_{N=-\infty}^{\infty} |N+1\rangle\langle N| \,, \qquad e^{i\hat{\theta}}|\theta\rangle = \theta|\theta\rangle \,, \qquad |\theta\rangle = \sum_{N=-\infty}^{\infty} \frac{e^{-i\theta N}}{\sqrt{2\pi}} |N\rangle$$

- external voltage

$$\boxed{\hat{H} = 2eV\hat{N} - E_J \cos(\hat{\theta})}$$

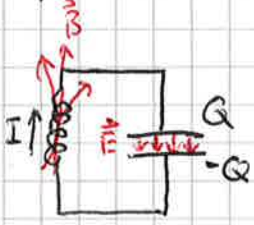
- Schrödinger equation

$$i\partial_t |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$$

solution: $|\Psi(t)\rangle = e^{iE_J \int_0^t dt' \cos(2eVt')} |\theta(t)\rangle$

with $\theta(t) = \theta_0 + 2eVt$ 2nd Josephson relation

ii) Structure the superconductor (circuit QED)



$$\boxed{\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}, = \Omega(\hat{a}^+\hat{a} + 1/2),}$$

capacitance Inductance

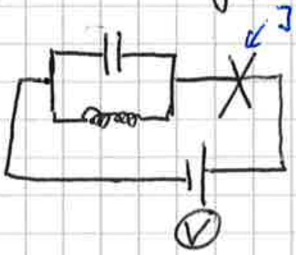
$[\hat{\Phi}, \hat{Q}] = i$

impedance

$\Omega = \frac{1}{\sqrt{LC}}$, $Z = \sqrt{\frac{L}{C}}$

$\hat{a} = i\hat{Q}\sqrt{\frac{Z}{2}} + \frac{\hat{\Phi}}{\sqrt{2Z}}$, $[\hat{a}, \hat{a}^+] = 1$

iii) Combining the two



Cooper pairs pick up a phase when tunneling

$$\hat{H} = \Omega\hat{a}^+\hat{a} + 2eV\hat{N} - E_J \cos(\hat{\theta} + 2e\hat{\Phi})$$

- Photo-assisted Cooper pair tunneling [9]

$2eV = k \cdot \Omega$

Cooper pair can tunnel by emitting k photons

- Von-Neumann equation

$$\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}]$$

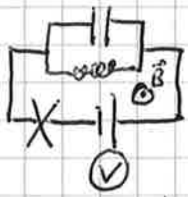
photons

solution: $\hat{\rho} = \hat{\rho}_r \otimes |\theta(t)\rangle\langle\theta(t)|$ with $\theta(t) = \theta_0 + 2eVt$

$\partial_t \hat{\rho}_r = -i[\hat{H}(t), \hat{\rho}_r]$, $\boxed{\hat{H}(t) = \Omega\hat{a}^+\hat{a} - E_J \cos(2eVt + 2e\hat{\Phi} + \theta_0)}$

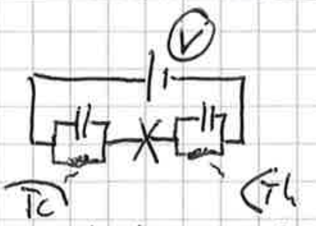
$2e\hat{\Phi} = 2\lambda(\hat{a} + \hat{a}^+)$, $\lambda = \sqrt{\frac{e^2 Z}{2}}$

II Quantum Thermal Machines



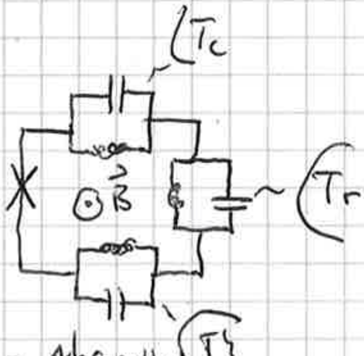
- work extractor [5]

Experiment [9]



- Heat engine [6]
- thermometer [7]

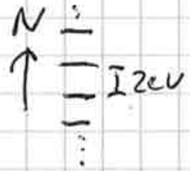
Experiment [10]



- Absorption refrigerator [8]

i) Work extractor

- System: $\hat{H}_S = \Omega \hat{a}^\dagger \hat{a}$
- Work storage device: $\hat{H}_W = 2eV\hat{N}$
- Coupling: $\hat{V} = -E_J \cos[\hat{\theta} + 2\lambda(\hat{a} + \hat{a}^\dagger)]$



Goal: Extract energy from system into work storage device
Work accessible through electrical current

$$\hat{P} = -i[\hat{H}_W, \hat{H}] = -2eVE_J \sin[\hat{\theta} + 2\lambda(\hat{a}+\hat{a}^\dagger)] = V \cdot \hat{I}$$ electrical current

Rotating wave approximation (RWA)

1. Go into rotating frame $\hat{U} = e^{i[\Omega \hat{a}^\dagger \hat{a} + 2eV\hat{N}]t}$

$$\hat{H} = -\frac{E_J}{2} e^{i(\hat{\theta} + 2eVt)} \underbrace{\exp\left[2i\lambda\left(\hat{a}^\dagger e^{i\Omega t} + \hat{a} e^{-i\Omega t}\right)\right]} + H.c.$$

$$= \sum_{l=0}^{\infty} i^l (\hat{a}^\dagger)^l \hat{A}(l) e^{il\Omega t} + \sum_{l=1}^{\infty} i^l \hat{A}(l)\hat{a}^l e^{-il\Omega t}$$

with $\hat{A}(l) = (2\lambda)^l e^{-2\lambda^2} \sum_{n=0}^{\infty} \frac{n!}{(n+l)!} L_n^{(l)}(4\lambda^2) |n\rangle\langle n|$, $\hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle$

generalized Laguerre polynomial

2. Set $2eV = k\Omega$ and drop all terms that oscillate in time

$$\Rightarrow \hat{V}_k = -\frac{E_J}{2}\left[i^k \hat{A}(k)\hat{a}^k e^{i\hat{\theta}} + (-i)^k (\hat{a}^\dagger)^k \hat{A}(k) e^{-i\hat{\theta}}\right]$$

tunneling Cooper pair absorbs k photons

$[\hat{V}_k, \hat{H}_S + \hat{H}_W] = 0 \Rightarrow$ All energy removed from system goes into work storage device

$\hat{\theta} \to 2eVt + \theta_0$: Work storage device in phase eigenstate!

Experimental Parameters:

- Voltage V (in situ)
- Magnetic field $\vec{B} \to \theta_0$ (in situ)
- Impedance $Z \to \lambda$ (fixed during experiment)
- Josephson Energy E_J (in situ)

i a) Gaussian states

$$\hat{\rho}_G = \hat{D}(\alpha)\, \hat{S}(\xi)\, \hat{\rho}_\beta\, \hat{S}^\dagger(\xi)\, \hat{D}^\dagger(\alpha)$$

· Thermal state $\hat{\rho}_\beta = \dfrac{e^{-\beta\Omega \hat{a}^\dagger \hat{a}}}{\mathrm{Tr}\{e^{-\beta\Omega\hat{a}^\dagger\hat{a}}\}}$

· Squeezing: $\hat{S}(\xi) = e^{\frac{1}{2}[\xi^* \hat{a}^2 - \xi(\hat{a}^\dagger)^2]}$, $\hat{S}(-\xi) = \hat{S}^\dagger(\xi) = \hat{S}^{-1}(\xi)$

· Displacement: $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$, $\hat{D}(-\alpha) = \hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha)$

Maximal amount of work can be extracted by undoing squeezing and displacement

$$\lambda \ll 1 \; : \quad \hat{A}(k) \to \frac{(2\lambda)^k}{k!}$$

$$U(t) = e^{-iV_k t} = \begin{cases} \hat{D}(\lambda E_J t e^{-i\theta_0}) & \text{for } k=1 \\ \hat{S}(2i\lambda^2 E_J t e^{-i\theta_0}) & \text{for } k=2 \end{cases} \quad \text{(rotating frame)}$$

Maximal amount of work is extracted by setting:

first: $k=1$, $2eV=\Omega$, $\theta_0 = \pi - \arg(\alpha)$ for time $\tau_\alpha = \dfrac{|\alpha|}{\lambda E_J}$

then: $k=2$, $2eV=2\Omega$, $\theta_0 = -\frac{\pi}{2} - \arg(\xi)$ for time $\tau_\xi = \dfrac{|\xi|}{2\lambda^2 E_J}$

ib) Fock state $|n\rangle$

For $2eV = n\Omega$, $\hat{V}_n$ connects the states

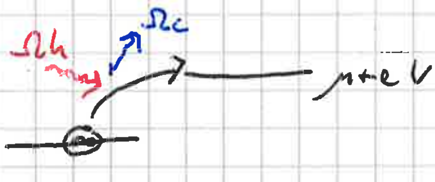
$$|0\rangle \leftrightarrow |n\rangle \not\leftrightarrow |2n\rangle \leftrightarrow \cdots$$

choose $4\lambda^2 = x_n$ s.t. $L_n^{(n)}(x_n) = 0$

$$\Rightarrow \hat{V}_n = -\frac{\pi}{2\tau_n} e^{i\theta} i^n |0\rangle\langle n| + \text{H.c.}, \qquad \tau_n = \frac{\pi}{E_J}\sqrt{\frac{n!}{(x_n)^n}}\, e^{x_n/2}$$

All energy can be extracted as work by evolving for time τ_n

ii) Heat Engine $\boxed{2eV = \Omega_h - \Omega_c}$



$$\hat{H} = \Omega_c \hat{a}_c^\dagger \hat{a}_c + \Omega_h \hat{a}_h^\dagger \hat{a}_h - E_J \cos\left[2eVt + 2\lambda_c(\hat{a}_c^\dagger + \hat{a}_c) + 2\lambda_h(\hat{a}_h^\dagger + \hat{a}_h)\right]$$

$$\hat{P} = -\partial_t \hat{H} = -2eV E_J \sin\left[2eVt + 2\lambda_c(\hat{a}_c^\dagger + \hat{a}_c) + 2\lambda_h(\hat{a}_h^\dagger + \hat{a}_h)\right]$$

$$\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}] + \mathcal{L}_c \hat{\rho} + \mathcal{L}_h \hat{\rho}$$

$$\mathcal{L}_\alpha = \kappa_\alpha \left\{ n_B^\alpha \mathcal{D}[\hat{a}_\alpha^\dagger] + [1 + n_B^\alpha]\mathcal{D}[\hat{a}_\alpha] \right\}$$

Bose-Einstein Distribution

$$\mathcal{D}[\hat{A}]\hat{\rho} = \hat{A}\hat{\rho}\hat{A}^\dagger - \frac{1}{2}\{\hat{A}^\dagger\hat{A}, \hat{\rho}\}, \qquad n_B^\alpha = \frac{1}{e^{\Omega_\alpha / k_B T_\alpha} - 1}$$

Rotating frame $\hat{U} = e^{i[\Omega_c \hat{a}_c^\dagger \hat{a}_c + \Omega_h \hat{a}_h^\dagger \hat{a}_h]t}$

$$\hat{H} = \frac{E_J}{2}\left[\hat{a}_c^\dagger \hat{A}_c(1) \hat{A}_h(1) \hat{a}_h + \hat{a}_h^\dagger \hat{A}_h(1) \hat{A}_c(1) \hat{a}_c\right]$$

$$\xrightarrow{\lambda_\alpha \ll 1} g(\hat{a}_c^\dagger \hat{a}_h + \hat{a}_h^\dagger \hat{a}_c), \qquad g = 2\lambda_c \lambda_h E_J$$

$$\hat{P} = -i\,2eV \frac{E_J}{2}\left[\hat{a}_c^\dagger \hat{A}_c(1) \hat{A}_h(1) \hat{a}_h - \hat{a}_h^\dagger \hat{A}_h(1) \hat{A}_c(1) \hat{a}_c\right],$$

$$\xrightarrow{\lambda_\alpha \ll 1} -i\,2eV g(\hat{a}_c^\dagger \hat{a}_h - \hat{a}_h^\dagger \hat{a}_c)$$

Time evolution of photons in cavity

$$\partial_t \langle \hat{a}_h^\dagger \hat{a}_h \rangle = \underbrace{-i\langle [\hat{a}_h^\dagger \hat{a}_h, \hat{H}] \rangle}_{P/2eV} + \underbrace{\mathrm{Tr}\{\hat{a}_h^\dagger \hat{a}_h \mathcal{L}_h \hat{\rho}\}}_{J_h/\Omega_h}$$

Steady state : $\partial_t \hat{\rho} = 0$

$$J_h = \frac{\Omega_h}{2eV} P \qquad J_c = -\frac{\Omega_c}{2eV} P$$

Efficiency

$$\eta = \frac{P}{J_h} = \frac{2eV}{\Omega_h} = 1 - \frac{\Omega_c}{\Omega_h} \underset{P \geq 0}{\leq} 1 - \frac{T_c}{T_h}$$

Otto efficiency

Entropy production

$$\Sigma = -\frac{J_h}{T_h} - \frac{J_c}{T_c} = \frac{P}{2eV}\left(\frac{\Omega_c}{T_c} - \frac{\Omega_h}{T_h}\right) \geq 0$$

$$P \geq 0 \quad \text{for} \quad \frac{\Omega_c}{T_c} \geq \frac{\Omega_h}{T_h} \iff n_B^h \geq n_B^c$$

III Conclusions

1. Superconductors are promising for quantum technology because the low-energy degrees of freedom can be engineered.

2. Photo-assisted Cooper pair tunneling provides a versatile platform for quantum thermodynamics