Second law of thermodynamics for quantum correlations

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Abstract

It is an established fact that quantum coherences have thermodynamic value. The natural question arises, whether other genuine quantum properties such as entanglement can also be exploited to extract thermodynamic work. In the present analysis, we show that the ergotropy can be expressed as a function of the quantum mutual information, which demonstrates the contributions to the extractable work from classical and quantum correlations. More specifically, we analyze bipartite quantum systems with locally thermal states, such that the only contribution to the ergotropy originates in the correlations. Our findings are illustrated for a two-qubit system collectively coupled to a thermal bath.

Preliminaries

For a quantum system described by $H=\sum_{i=1}^{d}\varepsilon_i\left|\varepsilon_i\right\rangle\left\langle\varepsilon_i\right|$ and quantum state $\rho=\sum_{j=1}^{d}r_j\left|r_j\right\rangle\left\langle r_j\right|$, such that $\varepsilon_i\leq\varepsilon_{i+1}$ and $r_j\geq r_{j+1}$. The ergotropy [1] is calculated by performing an optimization over all possible unitary operations to achieve a final state that has the minimum average energy with respect to H,

$$\mathcal{E}(\rho)=\mathrm{tr}\left\{H\rho\right\}-\min_{U}\left\{\mathrm{tr}\left\{HU\rho U^{\dagger}\right\}\right\}=\mathrm{tr}\left\{H(\rho-P_\rho)\right\}, \tag{1}$$

where $P_\rho\equiv\sum_k r_k\left|\varepsilon_k\right\rangle\left\langle\varepsilon_k\right|$ is called the *passive state*. An equivalent expression is the following

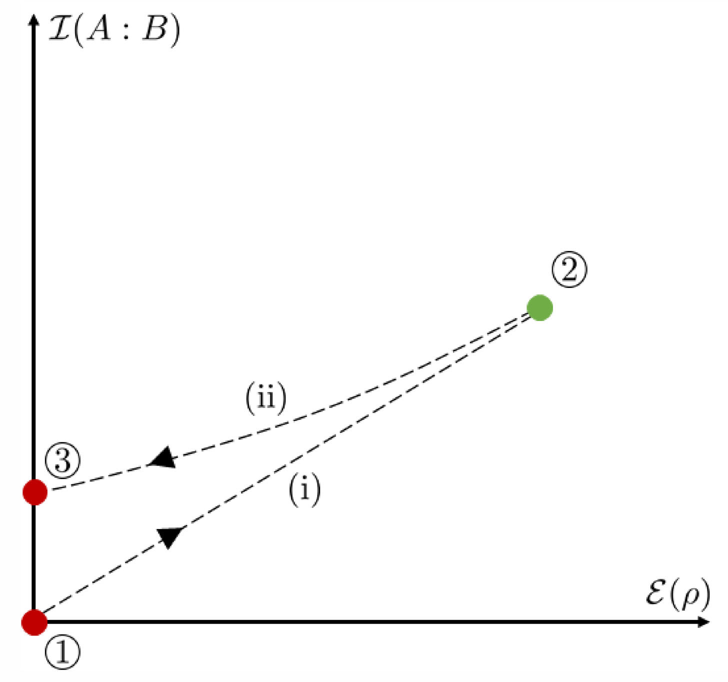
$$\mathcal{E}(\rho)=\sum_{i,j}r_j\varepsilon_i\left(\left|\left\langle r_j\middle|\varepsilon_i\right\rangle\right|^2-\delta_{ij}\right). \tag{2}$$

• **Goal:** relating the ergotropy to the correlations present in a quantum state, and quantified through the quantum mutual information,

$$\mathcal{I}(A:B)\equiv S(\rho_A)+S(\rho_B)-S(\rho), \tag{3}$$

where $S(\rho_i)=-\mathrm{tr}\left\{\rho_i\ln(\rho_i)\right\}$ denotes the von Neumann entropy, and ρ is the quantum state of $\mathcal{S}$.

Ergotropy from correlations



①: $t=-\tau_0$, $\rho(-\tau_0)=\rho_A(-\tau_0)\otimes\rho_B(-\tau_0)$

②: $t=0$, ρ

③: $t=\tau$, P_ρ

• *Process (i)*: **Create/sustain correlations**

$$(\forall t\in[-\tau_0,0]);\quad H(t)=H_A\otimes\mathbb{I}_B+\mathbb{I}_A\otimes H_B+H_I(t). \tag{4}$$

• *Process (ii)*: **Unitary work extraction**

$$(\forall t\in[0,\tau]);\quad H(t)=H_A\otimes\mathbb{I}_B+\mathbb{I}_A\otimes H_B+\Gamma(t). \tag{5}$$

$\Longrightarrow$ *Results*:

$$\beta\,\mathcal{E}(\rho)=\mathcal{I}(A:B)-D(P_\rho\|\rho_A\otimes\rho_B), \tag{6}$$

$$\beta\,\mathcal{E}(\rho)\leq\mathcal{I}(A:B). \tag{7}$$

Given the bound ergotropy $\mathcal{E}_b(\rho)=\mathrm{tr}\left\{(P_\rho-P_\rho^{\mathrm{th}})\right\}$, we get

$$\beta\left(\mathcal{E}(\rho)+\mathcal{E}_b(\rho)\right)\leq\mathcal{I}(A:B). \tag{8}$$

For the multipartite case, we have

$$\beta\,\mathcal{E}(\rho)\leq\mathcal{I}(A_1:A_2:\ldots:A_k). \tag{9}$$

$$\beta\,N\left(\mathcal{E}(\rho)+\mathcal{E}_b(\rho)\right)\leq N\,\mathcal{I}(A_1:A_2:\ldots:A_k), \tag{10}$$

Illustration

• We focus on X-states

$$\rho(t)=\begin{pmatrix}\rho_{11}&0&0&\rho_{14}\\0&\rho_{22}&\rho_{23}&0\\0&\rho_{23}^{*}&\rho_{33}&0\\\rho_{14}^{*}&0&0&\rho_{44}\end{pmatrix}. \tag{11}$$

• *Collective dissipation model* [2]:

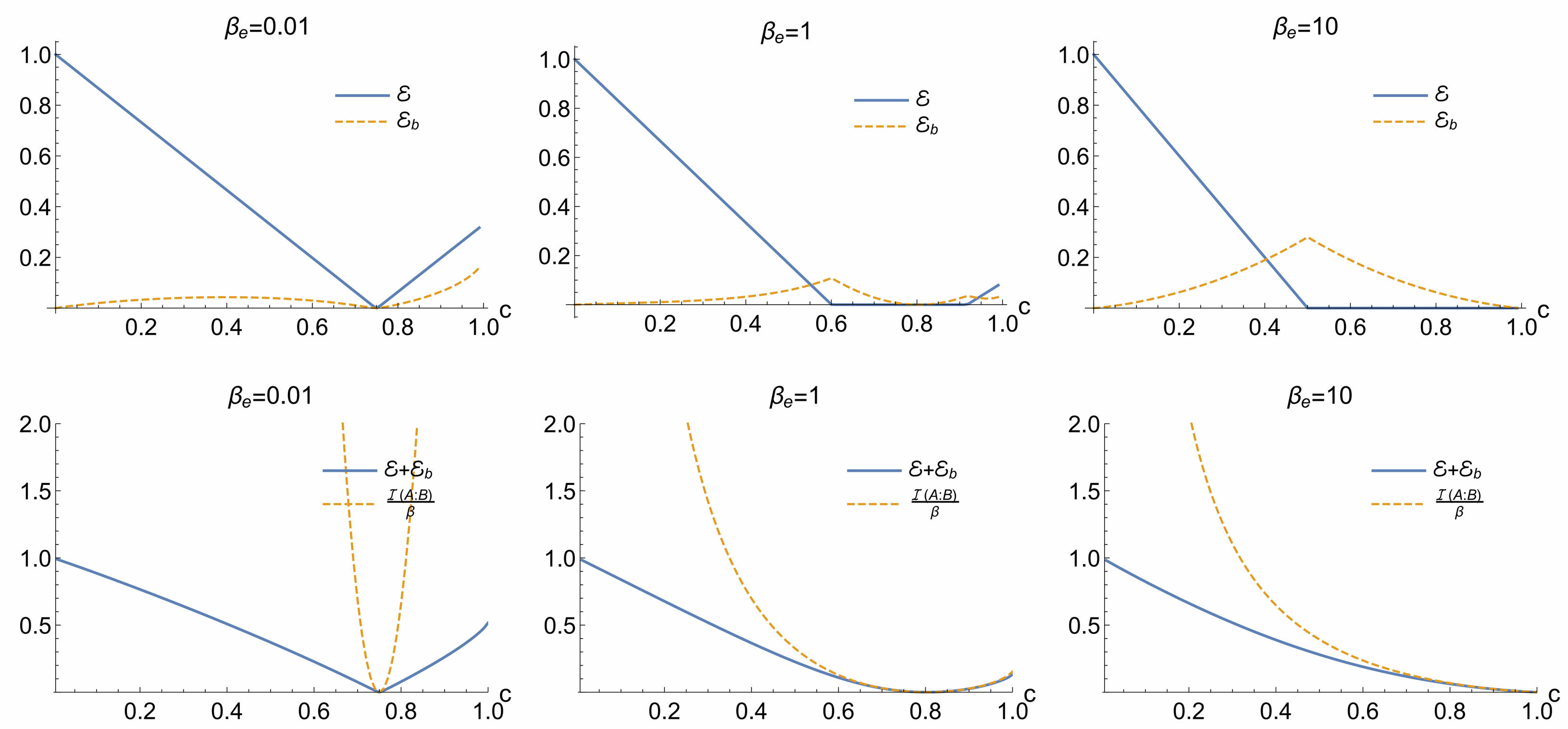
$$\frac{\partial\rho}{\partial t}=-\frac{i}{\hbar}[(H_0+H_d),\rho]+\mathcal{D}_-(\rho)+\mathcal{D}_+(\rho)=\mathcal{L}(\rho), \tag{12}$$

where $H_0=\hbar\omega(\sigma_1^+\sigma_1^-+\sigma_2^+\sigma_2^-)$ and $H_d=\hbar f(\sigma_1^+\sigma_2^-+\sigma_2^+\sigma_1^-)$ are the self-Hamiltonian of the whole system and the interaction Hamiltonian between the qubits, respectively. Additionally,

$$\begin{aligned}\mathcal{D}_-(\rho)&=\sum_{i,j=1}^{2}\gamma_{ij}\,(\bar{n}+1)(\sigma_j^-\rho\sigma_i^+-\frac{1}{2}\{\sigma_i^+\sigma_j^-,\rho\}),\\\mathcal{D}_+(\rho)&=\sum_{i,j=1}^{2}\gamma_{ij}\,\bar{n}(\sigma_j^+\rho\sigma_i^--\frac{1}{2}\{\sigma_i^-\sigma_j^+,\rho\}).\end{aligned} \tag{13}$$

Here, $\bar{n}=[\exp(\beta_e\omega)-1]^{-1}$ is the mean number of photons at the temperature of the environment β_e, and γ_{ij} are the spontaneous decay rates.

• *Plots*:



Concluding remarks

• *Bound on the average power*:

$$\mathcal{P}(\rho)\equiv\frac{\mathcal{E}(\rho)}{\tau}. \tag{14}$$

$$\Longrightarrow\ \mathcal{P}(\rho)\leq\mathcal{I}(A:B)\frac{G(\Omega,\Lambda)}{\hbar\beta\,\mathcal{L}\left(\rho,P_\rho\right)}. \tag{15}$$

We have,

$$G(\Omega,\Lambda)=\min\left\{\Omega,\sqrt{\left(\Omega+\langle\Lambda\rangle_0+\mathrm{tr}\left\{H_{AB}\right\}\right)\left(\Omega-\langle\Lambda\rangle_0\right)}\right\}, \tag{16}$$

such that $\Gamma(t)=\dot{\varphi}(t)\,\exp\left(-iHt/\hbar\right)\Lambda\,\exp\left(iHt/\hbar\right)$ and $\langle\Lambda\rangle_0=\mathrm{tr}\left\{\rho\Lambda\right\}$.

• Extracting thermodynamic work from a scrambled state [3,4].

• A first step towards a thermodynamic description of quantum computers [5].

References

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