Lecture notes: Quantum information in quantum thermal machines

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I. THE THREE-LEVEL MASER AS A QUANTUM THERMAL MACHINE

A thermal machine is a device that harnesses the flow of heat for the purpose of performing a task. Many of these machines are milestones inventions of the industrial age; perhaps most famously engines, heat pumps and refrigerators. However, such machines can also be considered on scales in which quantum theory is relevant. The earliest quantum version of a thermal machine was put forward by Scovil and Schulz-DuBois in 1959 [1] and was based on a re-interpetation of the so-called three-level maser (or amplifier) as a heat engine. Later, in 1967, it was found that by operating the process "in reverse", the three-level maser can be interpreted also as a refrigerator [2]. Although these earliest quantum thermal machines are neither the most fascinating nor the most realistic, they serve as simple introductions to the modus operandi of thermal machines in the quantum regime.

Consider a three-level quantum system (qutrit) with a ground state $|1\rangle$ and first and second excited states $|2\rangle$ and $|3\rangle$ respectively. In a qutrit, three different transitions are possible, namely $|1\rangle \leftrightarrow |2\rangle$, $|2\rangle \leftrightarrow |3\rangle$ and $|1\rangle \leftrightarrow |3\rangle$. We denote the energies associated with these transitions as E_c, E_h and E_r. Note that we require $E_r = E_c + E_h$. Each transition is separately coupled to a thermal bath of temperature T_c, T_h and T_r respectively. Here, the subscripts $\{c, r, h\}$ indicate one bath is cold, another is at an intermediate temperature ("room temperature") and the final one is hot. We therefore take $T_c < T_r < T_h$. The system is illustrated in Figure 1.

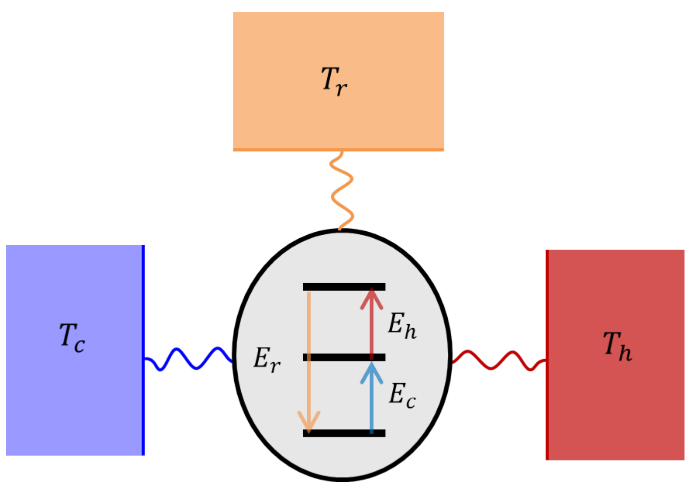

FIG. 1. The three-level maser as a refrigerator. Heat flows into the system from the cold bath and the hot bath and gets dumped into the room temperature bath.

In order to achieve refrigeration, the main idea is for the cold bath and the hot bath to each supply a quanta of energy to the qutrit, which is then dumped into the room bath. Let us identify the conditions under which this intuition works. It is instructive to momentarily remove the cold bath and note that the population ratios between the different energy levels then equilibrate to

$$\frac{q_3}{q_1} = e^{-E_r/T_r} \qquad \text{and} \qquad \frac{q_3}{q_2} = e^{-E_h/T_h}. \tag{1}$$

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For simplicity, we have set $\hbar = k_B = 1$ and denoted the populations as $q_i = \langle i|\rho|i\rangle$ for $i \in \{1,2,3\}$ and ρ being the state of the qutrit. These ratios imply that the population ratio associated to the cold transition becomes $\frac{q_2}{q_1} = e^{\frac{E_h}{T_h} - \frac{E_r}{T_r}}$. It is then a handy trick to define a virtual temperature, denoted T_v, which allows us to write this ratio in a manner similar to (1), namely $\frac{q_2}{q_1} = e^{-E_c/T_v}$. To this end, one must define the virtual temperature as

$$T_v = \frac{E_r - E_h}{E_r/T_r - E_h/T_h}, \tag{2}$$

where we have used that $E_c = E_r - E_h$. Now, if we were to re-connect the cold bath to our system, the transition $|1\rangle \leftrightarrow |2\rangle$ is activated and the system is perturbed out of its equilibrium state. The qutrit will then exchange excitations with the cold bath. The tendency is to *increase the temperature of this transition whenever the cold bath is hotter than the virtual temperature*. Thus, the condition for cooling becomes $T_c > T_v$. To satisfy it, one must suitably choose the energies and temperatures. Indeed, viewing quantum thermal machines through the perspective of a virtual qubit is many times useful for identifying conditions for cooling (refrigerator), heating (heat pump) and work creation (heat engine) [3].

This shows that the quantum thermal machine is indeed acts as a refrigerator. However, how well does it perform the cooling? Two quantities are particularly relevant to be aware of. The first one is the cooling power, i.e. the heat current[1] that flows out of the cold bath and into the system. This tells us how much cooling is going on, but nothing about the work cost paid to achieve it. This is addressed by the second quantity, called the coefficient of performance (COP)[2], defined as

$$\epsilon_{\text{fridge}} \equiv \frac{\text{heat current out of the cold bath}}{\text{heat current out of the hot bath}}. \tag{3}$$

Note that one can simultaneously have a large cooling power and a small COP, and vice versa. Since the heat currents are proportional (with a factor that is not of our interest) to the respective transition energies, we may also write the COP as $\epsilon_{\text{fridge}} = \frac{E_c}{E_h}$. We can now compute a bound on the COP that is independent of the transition energies. To this end, we first reforulate the cooling condition as

$$T_c > T_v \overset{(2)}{\Leftrightarrow} T_c > \frac{T_r T_h (E_r - E_h)}{E_r T_h - E_h T_r} \overset{E_r = E_c + E_h}{\Leftrightarrow} \frac{E_c}{E_h} < \frac{T_c (T_h - T_r)}{T_h (T_r - T_c)}. \tag{4}$$

In the right-most expression we have obtained an upper bound on the COP in terms of the bath temperatures. This is precisely the well-known Carnot efficiency[3]. Notably, in the limit of $T_h \to \infty$ it takes its more recognisable form $\epsilon_{\text{fridge}} \leq \frac{T_c}{T_r - T_c}$. The bound is saturated when $T_c = T_v$, which means that the refrigerator has the largest COP when its cooling power vanishes.

II. THREE QUBIT ABSORPTION REFRIDGERATOR

A more complicated, but also more realistic, quantum absorption refrigerator is one based on three qubits. In contrast to the three-level maser, the couplings between the baths and the desired transitions are more straightforward when each qubit is separately connected to a bath. Here, we follow the refrigerator proposed by Linden et. al. [4].

A. Scenario

Consider three qubits whose respective excited states have energy E_c, E_r and E_h, tuned in such a way that $E_r = E_c + E_h$ (and $E_c \neq E_h$). The free Hamiltonian is therefore

$$H_{\text{free}} = E_c |1\rangle\langle 1|_c \otimes \mathbb{1}_r \otimes \mathbb{1}_h + \mathbb{1}_c \otimes E_r |1\rangle\langle 1|_r \otimes \mathbb{1}_h + \mathbb{1}_c \otimes \mathbb{1}_r \otimes E_h |1\rangle\langle 1|_h. \tag{5}$$

By construction, the free Hamiltonian has two energy-degerate eigenstates, $|010\rangle$ and $|101\rangle$. Consider now that we let the three qubits interact weakly over a Hamiltonian H_{int}. Due to weak coupling, the non-resonant couplings will be surpressed (roughly corresponding to their energy difference) and the resonant transition will dominate. We focus on this regime and write the Hamiltonian as

$$H_{\text{int}} = g \left(|010\rangle\langle 101| + |101\rangle\langle 010| \right), \tag{6}$$

where $g \geq 0$ is the strength of the interaction. Thus, weak coupling amounts to taking $\forall i : g \ll E_i$. The goal is to use the room qubit and the hot qubit to cool the cold qubit. This three-qubit refridgerator is illustrated in Figure 2.

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[1] Recall that the heat current is is the change in the dissipative energy, with a sign indicating whether it goes into the system or out of the system.
[2] It is not uncommon that the COP is refered to as the efficiency (of the refrigerator). This is somewhat misleading, since it can be larger than one.
[3] Recall that the Carnot efficiency can be derived immediately from the first and second law, namely $E_c + E_h - E_r = 0$ and $-E_c/T_c + E_r/T_r - E_h/T_h \geq 0$, where we have implicitely assumed that cooling takes place (notice the sign of the heat flow).

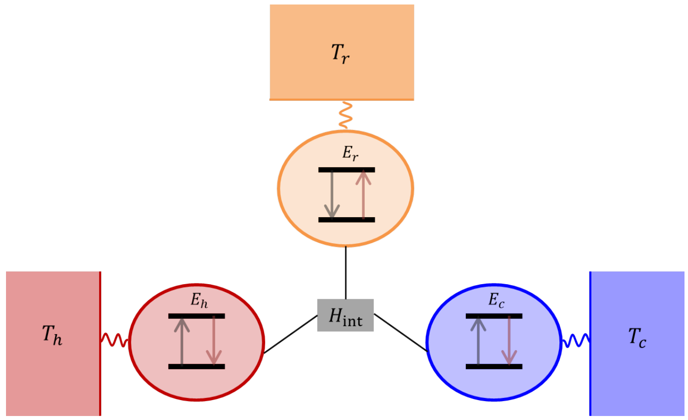

FIG. 2. Three qubits are weakly coupled over an interaction Hamiltonian and also individually coupled to respective baths. The aim is to tune the energies, temperatures and interaction in such a way that the room temperature qubit and the hot qubit can be used to cool the cold qubit.

B. Cooling conditions

Now, let us denote the state of the three qubits by ρ and consider its dynamics, as influenced both by the weak inter-qubit interactions and the system-environment interactions. We will work in the regime in which the Lindblad (or, LGKS) equation is valid. This requires us to make the main assumptions

- The system and the bath are initiated in a product state.
- The coupling between the system and the bath leaves the latter essentially unaffected. In other words, the bath is much "larger" than the system and the coupling between them is weak. This is known as the Born approximation. In practice, this means that the quantum state of the environment and system factors at all times.
- The baths are essentially memoryless, i.e. $\frac{d\rho}{dt}$ depends only on $\rho(t)$ and not the state at its past times. This means that any correlations within the bath are lost on a faster scale than that which influences the evolution of ρ. This is called the Markov assumption.

In such a Lindblad setting, a particularly simple heurestic, known as a reset model, is handy:

$$\frac{d\rho}{dt} = \underbrace{-i[H_{\text{free}} + H_{\text{int}}, \rho]}_{\text{closed evolution}} + \underbrace{\sum_{i\in\{c,r,h\}} \gamma_i \left(\tau_i \otimes \text{tr}_i\left(\rho\right) - \rho\right)}_{\text{dissipative evolution}}, \tag{7}$$

where $\tau_i = \frac{1}{1+e^{-E_i/T_i}}\left(|0\rangle\langle 0| + e^{-E_i/T_i}|1\rangle\langle 1|\right)$ is the thermal state and $\text{tr}_i\left(\rho\right)$ is the partial trace over system i. Thus, the intuition is that the i'th qubit is lost to the i'th bath and replaced with a thermal state at rate γ_i. Notice that the reset equation is trace-preserving, i.e. that $\text{tr}\,\frac{d\rho}{dt} = 0$ and it may be viewed as a special instance of a Lindblad equation.

We will be interested in the steady-state behaviour of the system, i.e. in the limit $t \to \infty$ when $\frac{d\rho}{dt} = 0$. Thus, to obtain the steady-state of the system, we must solve inhomogenous linear matrix equation. It is handy to write the master equation on the form $0 = \mathbf{A}\vec{x} + \mathbf{b}$ where $\vec{x}$ is a vector of all the variables of the density matrix ρ. The square matrix $\mathbf{A}$ and the column matrix $\mathbf{b}$ depend on the physical parameters $(T_c, T_r, T_h, E_c, E_r, E_h, \gamma_c, \gamma_r, \gamma_h, g)$. In this manner, the maser equation can be solved as $\vec{x} = -\mathbf{A}^{-1}\mathbf{b}$. Notice, however, that the equation is considerably simplified by the fact that interaction Hamiltonian only can generate coherence in the subspace of the resonant transitions, i.e. the state is at all times of the form

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & \rho_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{55} & 0 & 0 & 0 \\ 0 & 0 & c^* & 0 & 0 & \rho_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{88} \end{pmatrix}. \tag{8}$$

The explicit form of the steady-state ρ^∞ is therefore straightforward to calculate but cumbersome to spell out in detail. Full details are found in [5]. Here, we only highlight some key properties of the solution. Firstly, it can be written on the form

$$\rho^\infty = \tau_c \otimes \tau_r \otimes \tau_h + \alpha\sigma, \tag{9}$$

where α is a dimensionless parameter that depends on all the physical parameters $(T_c, T_r, T_h, E_c, E_r, E_h, \gamma_c, \gamma_r, \gamma_h, g)$ and σ is a trace-zero matrix that only has a single coherence term (in analogy with (8)). Let us now inspect the individual states of the qubits:

$$\rho_i^\infty = \tau_i + \alpha \operatorname{tr}_{\bar{i}} \sigma, \tag{10}$$

where $\bar{i}$ denotes partial trace over the two qubits that are not i. For a matrix of the form (8), it is easily seen that all three reduced states are diagonal. In fact, they are not only diagonal but turn out to be proportional to the Pauli Z matrix. Therefore, the state of our first (target) qubit becomes $\rho_c^\infty = \tau_c + \frac{\alpha}{\gamma_c}\bar{\sigma}_Z$, where $\bar{\sigma}_Z$ is just the Pauli Z matrix multiplied by a positive constant. Importantly, notice that ρ_1^∞ is effectively a thermal state - because every diagonal qubit state is a thermal state corresponding to some suitable ratio $\frac{E}{T}$. Thus, the machine is effectively cooling the cold qubit if this thermal state corresponds to a temperature lower than that of the bath, which is manifest in τ_c. In other words, the machine needs to increase the ground state population of the cold qubit. This is immediately seen to be equivalent to the condition $\alpha > 0$. Of course, this cooling condition can be spelled out explicitly by inspecting the form of the steady-state solution. It turns out this becomes equivalent to

$$\alpha > 0 \Leftrightarrow \langle 0|\tau_c|0\rangle\langle 1|\tau_r|1\rangle\langle 0|\tau_h|0\rangle < \langle 1|\tau_c|1\rangle\langle 0|\tau_r|0\rangle\langle 1|\tau_h|1\rangle \Leftrightarrow e^{-E_c/T_c}e^{-E_h/T_h} > e^{-E_r/T_r} \Leftrightarrow \frac{E_c}{E_h} < \frac{T_c\left(T_h - T_r\right)}{T_h\left(T_r - T_c\right)}. \tag{11}$$

This can be thought of as the total (separate equilibrium) population in $|010\rangle$ being smaller than in $|101\rangle$, thus favouring a transition $|101\rangle \to |010\rangle$ which entails an increase of the ground state population of the cold qubit. Also, one may immediately notice the similarity to the result (4) obtained for the three-level maser. Let us properly show that this indeed captures the COP of this machine. To this end, consider the variation of a single qubit state

$$\delta\rho_i(t) = \rho_i(t+\delta t) - \rho_i(t) = \left[\rho_i(t) + \delta t\gamma_i\left(\tau_i - \rho_i(t)\right)\right] - \rho_i(t) = \delta t\gamma_i\left(\tau_i - \rho_i\right). \tag{12}$$

Using this, we have that the variation of the energy is

$$\operatorname{tr}\left(H_i\delta\rho_i(t)\right) = \gamma_i\delta t \operatorname{tr}\left(H_i\left(\tau_i - \rho_i\right)\right). \tag{13}$$

The heat current (into the system) is therefore

$$J_i = \gamma_i \operatorname{tr}\left(H_i\left(\tau_i - \rho_i(t)\right)\right). \tag{14}$$

With our free Hamiltonian, the expression for the heat current reduces to

$$J_i = E_i\gamma_i\langle 1|\left(\tau_i - \rho_i(t)\right)|1\rangle. \tag{15}$$

Since we know the form of the local state, we therefore have that

$$J_i = E_i\gamma_i\langle 1| \pm \frac{\alpha}{\gamma_i}\bar{\sigma}_Z|1\rangle \propto E_i. \tag{16}$$

Thus, the COP $\epsilon_{\text{fridge}} \equiv J_c/J_h$ is just the ratio E_c/E_h and the analogy with the three-level maser and the Carnot limit hold.

III. NONCLASSICALITY IN QUANTUM THERMAL MACHINES

What is so *quantum* about a quantum thermal machine, other than the fact that we have chosen to describe it with the formalism of quantum theory? This touches of one of the deeper questions that are relevant to the role of quantum information in quantum thermal machines. In the context of the three-qubit refrigerator, an interesting fact in this regard appears if we ask ourselves: *which type of steady state* (9) *achieves the best cooling?* It has been found that if we insist that the steady-state is separable, i.e. that it can be prepared by means of stochastically coordinated singe-qubit operations

$$\rho^\infty = \sum_\lambda p_\lambda \psi_c^{(\lambda)} \otimes \psi_r^{(\lambda)} \otimes \psi_h^{(\lambda)}, \tag{17}$$

where $\{p_\lambda\}$ is a probability distribution and $\psi_i^{(\lambda)}$ is a state, then we cannot achieve as much cooling as compared to the case when the state *cannot* be written on the form (17). In other words, at fix the properties of the cold qubit and the temperatures of the baths, we can find parameters (p_r, p_h, E_r, E_h, g) that allow us to cool more if the correspond steady state is entangled (does not admit the form (17)). Entanglement is therefore a resource for cooling. This was shown by numerical examination of the steady state (9) in [6].

A. Two-qubit entanglement engine

This leads us to an interesting question: to what extent can quantum thermal machines produce nonclassical phenomena? Let us consider the simplest thermal machine, namely one that consists only of two resonant qubits (each at excited energy E) that interact weakly via a flip-flop Hamiltonian $H_{\text{int}} = g\left(|0,1\rangle\langle 1,0| + |1,0\rangle\langle 0,1|\right)$. The free Hamiltonian is $H_{\text{free}} = E|1\rangle\langle 1| \otimes \mathbb{1} + \mathbb{1} \otimes E|1\rangle\langle 1|$. Each qubit is, once more, coupled to a bath at temperature T_h and T_c respectively. See Figure 3.

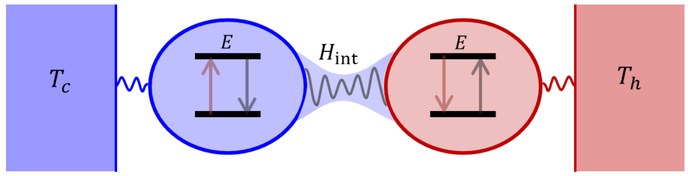

FIG. 3. Simplest entanglement engine: spontaneous thermal interactions and a heat current is used to create an entangled steady state between two qubits. Originally introduced in [7]

This time, we do not settle for a simple reset model and instead embrace a full-out Lindblad equation, which takes the form

$$\frac{\partial \rho}{\partial t} = \underbrace{-i\left[H_{\text{free}} + H_{\text{int}}, \rho\right]}_{\text{closed evolution}} + \underbrace{\sum_{i\in\{c,h\}} \Gamma_i^+ \left(L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}\right)}_{\text{dissipative evolution, excitations received}} + \underbrace{\sum_{i\in\{c,h\}} \Gamma_i^- \left(L_i^\dagger \rho L_i - \frac{1}{2}\{L_i L_i^\dagger, \rho\}\right)}_{\text{dissipative evolution, excitations lost}}. \tag{18}$$

The jump operators are local, i.e. they increase the energy of each qubit separately: $L_c = |1\rangle\langle 0| \otimes \mathbb{1}$ and $L_h = \mathbb{1} \otimes |0\rangle\langle 1|$. The system-bath couplings are determined by the energy E, the temperature of the bath T_i and the particle statistics. For sake of example, let us consider fermionic statistics, thus described by a Fermi-Dirac distribution $n_{\text{FD}}(E,T) = \left(1 + e^{E/T_i}\right)^{-1}$ for $i \in \{c,h\}$. Then, if the coupling strength between the system-bath interaction is γ_i, the system receives excitations at the rate $\Gamma_i^+ = \gamma_i n_{\text{FD}}(E,T_i)$ and loses excitations at the rate $\Gamma_i^- = \gamma_i\left(1 - n_{\text{FD}}(E,T_i)\right)$. This specifies our local Lindblad master equation. In analogy with the previously considered reset model, we can obtain the steady state by solving an inhomogenous linear matrix equation ($\frac{\partial\rho}{\partial t} = 0$). Since the production of entanglement is associate to a large heat current, it is instructive to consider the limiting case in which the cold bath is very cold ($T_c \to 0^+$) and the hot bath is very hot ($T_h \to \infty$). This considerably simplifies the system-bath couplings because $n_{\text{FD}}(E,0) = 0$ and $n_{\text{FD}}(E,\infty) = \frac{1}{2}$. The steady state is then found to be

$$\rho^\infty = \frac{1}{2N}\begin{pmatrix} \gamma_c\gamma_h t^2 + 2g^2 s^2 & 0 & 0 & 0 \\ 0 & 2g^2\gamma_c s & -2igt\gamma_c\gamma_h & 0 \\ 0 & 2itg\gamma_c\gamma_h & \gamma_c\left(\gamma_h t^2 + 2g^2 s\right) & 0 \\ 0 & 0 & 0 & 2g^2\gamma_c^2 \end{pmatrix}, \tag{19}$$

where $t = \gamma_c + \gamma_h$, $s = \gamma_c + 2\gamma_h$, and $N = t^2\left(4g^2 + \gamma_c\gamma_h\right)$.

B. Entanglement condition

Let us begin by proving that our machine can produce steady-state entanglement. Since it is a two-qubit system, we are in the exceptional situation in which we can give a necessary and sufficient condition for the existence of entanglement. To this end, we can use the positive partial transpose criterion, which states that a two-qubit state is entangled if and only if the operator ρ^{T_A} has a negative eigenvalue. Consider a state of the form

$$\begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & c & 0 \\ 0 & c^* & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{20}$$

After a partial transpose on the first qubit, the state now reads

$$\begin{pmatrix} \rho_{11} & 0 & 0 & c^* \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ c & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{21}$$

Three of its eigenvalues are necessarily positive. The fourth eigenvalue reads

$$\lambda = \frac{1}{2}\left(\rho_{11} + \rho_{44} - \sqrt{(\rho_{11} - \rho_{44})^2 + 4|c|^2}\right). \tag{22}$$

The entanglement condition $\lambda < 0$ can then easily be found to be equivalent to

$$|c|^2 > \rho_{11}\rho_{44}. \tag{23}$$

If we use our steady state (19), a this condition takes the form

$$g^2 < \frac{(2\gamma_h - \gamma_c)\,\gamma_h\,(\gamma_c + \gamma_h)^2}{2\,(\gamma_c + 2\gamma_h)^2}. \tag{24}$$

In the limit of small g, this criterion essentially reduces to the simple expression $\gamma_h > \gamma_c/2$. We conclude that it is possible to generate steady state entanglement in machines that only harness spontaneous thermal interactions.

C. Teleportation and negative temperature

The machine that we have considered is the simplest of its kind. It only involves two qubits, it has no work input or external control and the baths are classical (they are collections of thermal states). It is therefore rather remarkable that entanglement production is possible at all. However, a bit of inspection of (19) reveals that the entanglement is rather weak and noisy.

The claim that entanglement is weak and noisy has a subtle meaning. Indeed, it is not in general possible to objectively compare the magnitude of entanglement (even in the two-qubit case). The reason is that the set of monotones under local operations and classical communication admits only a partial, but not a global, order. This means that if f_A and f_B are two entanglement monotones (invariant under LOCC) then there exists situations in which $f_A(\psi_1) \leq f_A(\psi_2)$ and $f_B(\psi_2) \leq f_B(\psi_1)$, i.e. the question of "which state is more entangled" can only be answered relative to the choice entanglement monotone. However, this choice is many times rather arbitrary. It is therefore many times more interesting and meaningful to talk about entanglement in an *operational* manner, i.e. in terms of concrete tasks in which it serves as a resource to break classical limitations. It turns out that the entanglement of our steady state (19) is weak in the sense that it cannot be used for several paradigmatic tests of nonclassicality [10]. Here, we will focus on the task of quantum teleportation.

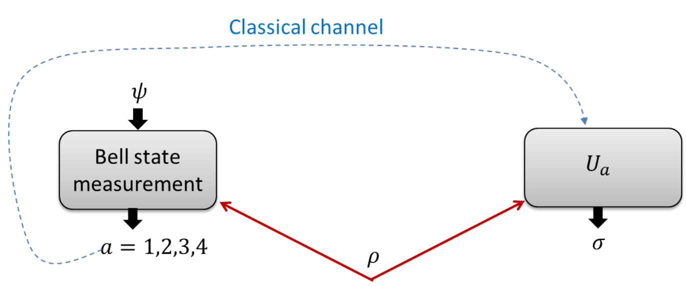

FIG. 4. Two parties share a state ρ as a resource for teleporting the state ψ from the left to the right. In a standard teleportation protocol, a Bell state measurement is followed by a classical communication and a correction unitary. The resulting state σ is the output of the protocol, which is meant to be as close to ψ as possible.

In quantum teleportation (see Figure 4), Alice wants to send Bob a qubit state $|\psi\rangle$. The available resource is a shared entangled state ρ and classical communication. The standard protocol for performing a teleportation task is to let Alice measure her share of ρ and her input state ψ in a basis of Bell states, i.e. she projects onto the four states $\{|00\rangle \pm |11\rangle, |01\rangle \pm |10\rangle\}$. The outcome is labeled a and sent off to Bob who uses this to perform a unitary transformation on his share of ρ. This leaves him in the state σ which is the final result of the teleportation protocol. Ideally, we want $\sigma = \psi$ but this is only possible if the shared state is maximally entangled, e.g. $\rho \sim |00\rangle + |11\rangle$. When perfect teleportation is not possible (nearly always), we can instead quantify the closeness between ψ and σ using the so-called fidelity of teleportation. This is defined as

$$f = \int d\psi \langle\psi|\sigma|\psi\rangle, \tag{25}$$

i.e. it is the fidelity averaged over all quantum states. The fundamental question for teleportation is therefore: *if we are given the two-qubit state ρ as our teleportation resource, what is the achievable fidelity of teleportation?*. The answer to this question was provided in 1999 [9]. The answer is that f is immediately related to the so-called *maximally entangled fraction*, F, through the formula $f = \frac{1+2F}{3}$ where

$$F(\rho) = \max_U \langle \psi^- | \left(\mathbb{1} \otimes U\right) \rho \left(\mathbb{1} \otimes U^\dagger\right) |\psi^-\rangle, \tag{26}$$

where $|\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$ and U is a qubit unitary. It is easily seen that every product state has $F = 1/2$ and therefore $f = 2/3$. Thus, the condition for nonclassical teleportation is

$$F(\rho) > \frac{1}{2} \Leftrightarrow \text{The entangled state } \rho \text{ is a resource for nonclassical teleportation.} \tag{27}$$

The maximisation over the unitary in (26) complicates the calculation of the maximally entangled fraction. However, since we are only interested in states of the form (20), a closed analytical expression is possible. If $F > 1/2$, then it can be calculated as

$$F = |c| + \frac{\rho_{22} + \rho_{33}}{2} > \frac{1}{2}. \tag{28}$$

However, a critical fact is that not all entangled two-qubit states satisfy this condition, i.e. there exists entangled states are not able to beat the classical limit. This follows by comparing the teleportation condtion (28) with the entanglement condition (23). In fact, it turns out that for every choice of machine parameters (bath temperatures, energy gap, system-bath coupling strength, strength of Hamiltonian) the entanglement produced in the machine's steady state (19) cannot be used to achieve teleportation! In other words, it is too weak to be useful for this type of quantum information processing.

Let us therefore as consider the possibility of boosting entanglement production by amending the elementary machine with somewhat more sophisticated resources. We focus on one such resource, namely the possibility of having a *negative temperature*. A negative temperature is an effect that is possible in quantum system and we have in fact already encountered a system in which this happens. When the three-level maser is run the engine mode, it relies on a process of population inversion, i.e. a process that populates excited states more than the ground states. This is what is mathematically meant by a negative temperature since such an effect appears in the thermal state $\tau = \frac{1}{1+e^{-E/T}} \left(|0\rangle\langle 0| + e^{-E/T}|1\rangle\langle 1|\right)$ when $T < 0$. Notice that a maximal population inversion, in which only the excited state is populated, corresponds to a temperature $T \to 0^-$. Thus, this is the "hottest" case. Imagine now that we allow the hot bath in our machine (3) to perform such an ideal population inversion. In analogy the previous, one can calculate the new steady state to be

$$\rho^{\infty,inv} = \frac{1}{N} \begin{pmatrix} 4g^2\gamma_h^2 & 0 & 0 & 0 \\ 0 & 4g^2\gamma_c\gamma_h & -2itg\gamma_c\gamma_h & 0 \\ 0 & 2itg\gamma_c\gamma_h & \gamma_c\gamma_h\left(4g^2 + t^2\right) & 0 \\ 0 & 0 & 0 & 4g^2\gamma_c^2 \end{pmatrix}. \tag{29}$$

Does this give us "stronger" entanglement than the elementary machine? The answer is yes[4]. Consider that we choose $\gamma_c = \gamma_h$ and $g = \frac{\sqrt{5}-1}{4}\gamma_c$. If one inserts this into the steady state (29) and then inserts that into the expression for the maximally entangled fraction, one finds $F = \frac{3+\sqrt{5}}{8} \approx 0.65$. This exceeds the classical limit of teleportation. We conclude that our enhanced quantum thermal machine is powerful enough to produce entanglement that is useful for this paradigmatic task of quantum information processing [10].

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[4] The main intuition is that the production of entanglement is known to be closely linked to the magnitude of the heat current [11]. A larger temperature gradient enables a larger heat current.

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