Quantum Energy Lines and the optimal output ergotropy problem

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Abstract

We study the possibility of conveying useful energy (work) along a transmission line that allows for a partial preservation of quantum coherence. As a figure of merit we adopt the maximum values that ergotropy, total ergotropy, and non-equilibrium free-energy attain at the output of the line for an assigned input energy threshold. When the system can be modelled in terms of Phase-Invariant Bosonic Gaussian Channels (BGCs), we show that coherent inputs are optimal. For generic BGCs which are not Phase-Invariant the problem becomes more complex and coherent inputs are no longer optimal. In this case, focusing on one-mode channels, we solve the optimization problem under the extra restriction of Gaussian input signals.

Bosonic Gaussian States

We consider the n-mode electromagnetic Hamiltonian $\hat{H} = \sum_{j=1}^{n} \frac{\hat{q}_j^2 + \hat{p}_j^2}{2} - \frac{n}{2}$, where $\hat{q}_j$ and $\hat{p}_j$ are the local position and momentum operators respectively, in order to get a more compact notation we define $\hat{r} = (\hat{q}_1, \hat{p}_1, ..., \hat{q}_n, \hat{p}_n)^T$.

The characteristic function of a bosonic quantum state $\hat{\rho}$ is $\chi(\hat{\rho}; x) = \mathrm{Tr}[\hat{\rho}\hat{D}(x)]$, where $x \in \mathbb{R}^2$ and $\hat{D}(x) = \exp[i\hat{r} \cdot x]$. A bosonic n-mode Gaussian state is a state whose characteristic function has a gaussian form $\chi(\hat{\rho}; x) = e^{-\frac{1}{4}x^T\sigma x + im\cdot x}$, where $m = \mathrm{Tr}[\hat{\rho}\hat{r}]$ is the statistical mean of the state and σ with entries $\sigma_{jk} = \mathrm{Tr}[\hat{\rho}\hat{r}_j, \hat{r}_k]$ is the *covariance matrix*. A Gaussian state can be uniquely identified by its first two statistical moments. Coherent states are a particular subclass of these states: they are defined as $\hat{D}(x)\,|0\rangle\,\langle 0|\,\hat{D}(x)^\dagger$, they have $m = x$ and $\sigma = I_{2n}$.

The majority of quantum states realizable in linear Quantum Optics are Gaussian states, since coherent, squeezed and thermal states are all Gaussian.

Bosonic Gaussian Channels (BGCs)

A Bosonic Gaussian Channel (BGC) acting on n_I input modes with n_O output modes transforms the characteristic function of any quantum state in the following way: $\chi(\Phi(\hat{\rho}); x) = \chi(\hat{\rho}; X^T x)\exp[-\frac{1}{4}x^T Y x + im\dot{x}]$ where $X \in \mathbb{R}^{2n_O \times 2n_I}$ and $Y \in \mathbb{R}^{2n_I \times 2n_I}$ is positive semidefinite and $Y \geq i(\gamma_{n_O} + X\gamma_{n_I}X^T)$ with $\gamma_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. So BGCs transform the first two statistical moments of an n-mode bosonic state in the following way $m' = Xm \qquad \sigma' = X\sigma X^T + Y$. BGCs send Gaussian states into Gaussain states.

Phase-Insensistive BGCs (PI-BGCs)

Among BGCs Phase-Insensitive Bosonic Gaussian Channels (PI-BGCs) represent a large and physically motivated class of operations, in particular the are defined as BGCs such that $\forall t \in \mathbb{R}$

$$\Phi(e^{-i\hat{H}_I}\hat{\rho}e^{i\hat{H}_I}) = e^{\mp i\hat{H}_O}\Phi(\hat{\rho})e^{\pm i\hat{H}_O},$$

where $\hat{H}_I$ is the n_I-mode input Hamiltonian while $\hat{H}_O$ is the n_O-mode output Hamiltonian. One mode PI-BGCs can model both attenuation and amplification processes subject to environmental thermal noise, for instance the noisy loss channel $\mathcal{L}_{\eta,N}$, the noisy amplification channel $\mathcal{A}_{\mu,N}$ and the additive noise channel $\mathcal{N}_N$ are all one-mode PI-BGCs. From a practical standpoint they represent idealized versions of optical amplifiers and optical fibers. Moreover multimode PI-BGCs can provide idealized versions of broadband communication in the optical regime.

Majorization

Given two quantum states $\hat{\rho}$ and $\hat{\xi}$, we say that $\hat{\rho}$ *majorizes* $\hat{\xi}$ if

$$\forall k \in \mathbb{N} \sum_{j=1}^{k} p_j(\hat{\rho}) \geq \sum_{j=1}^{k} p_j(\hat{\xi})$$

where we compared the spectra of the two states ordered in decreasing order.

Majorization is a partial order between quantum states, moreover several quantum state functional are Schur-convex, i.e. $f(\hat{\rho}) \geq f(\hat{\xi})$ if $\hat{\rho} \succ \hat{\xi}$, so majorization can be used to optimize various figures of merit.

Quantum Work Extraction

In Quantum Mechanics the average energy of a system that can be used depends on the work extraction protocol, in particular we study three different figures of merit which are widely used in Quantum Thermodynamics.

Ergotropy is the average maximum work that can be extracted via unitary operations:

$$\mathcal{E}(\hat{\rho}) = \mathrm{Tr}[\hat{\rho}] - \min_{\hat{V}} \mathrm{Tr}[\hat{V}\hat{\rho}\hat{V}^\dagger\hat{H}].$$

Total ergotropy is the regularized ergotropy of infinite copies of a system:

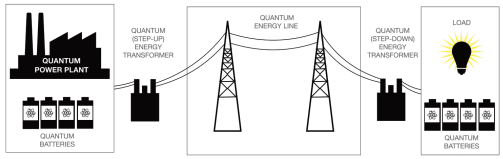
$$\mathcal{E}_{tot}(\hat{\rho}) = \lim_{n\to\infty} \frac{\mathcal{E}(\hat{\rho}^{\otimes n})}{n}.$$

Non-equilibrium free energy is the extractable energy through thermal operations using a thermal bath with inverse temperature β:

$$\mathcal{F}^\beta(\hat{\rho}) = \mathrm{Tr}[\hat{\rho}\hat{H}] - \beta^{-1}S(\hat{\rho}).$$

They are all Schur-convex functionals with states at fixed energy!

Quantum Energy Lines



We study the energy transport from a *quantum source* to a *quantum user*. We do so by focusing on noisy bosonic quantum channels that are widely used as information carriers, in particular we find the best possible protocols to preserve the erogtropy, the total ergotropy and the non-equilibrium free energy at the output of BGCs. Moreover our work is useful in order to maximize the energy storage of a bosonic Quantum Battery.

Results on PI-BGCs

Theorem 1 *For any $n \in \mathbb{N}$, $E \in \mathbb{R}^+$, $\hat{\rho}$ with $Tr[\hat{\rho}\hat{H}] \leq E$, for any n-mode PI-BGC Φ there exists a coherent state $\hat{\varphi}$ with $Tr[\hat{\varphi}] = E$ such that*

$$\mathcal{E}(\Phi(\hat{\varphi})) \geq \mathcal{E}(\Phi(\hat{\rho})) \qquad \mathcal{E}_{tot}(\Phi(\hat{\varphi})) \geq \mathcal{E}_{tot}(\Phi(\hat{\rho})) \qquad \mathcal{F}^\beta(\Phi(\hat{\varphi})) \geq \mathcal{F}^\beta(\Phi(\hat{\rho}))$$

Theorem 2 *For any $s, E \in \mathbb{R}^*$, $\hat{\rho}$ with $Tr[\hat{\rho}\hat{H}] \leq E$ and $S(\hat{\rho}) \geq s$, for any one-mode PI-BGC Φ there exists a displaced thermal state $\hat{\varphi}$ with $Tr[\hat{\varphi}\hat{H}]$ and $S(\hat{\varphi}) = s$ such that*

$$\mathcal{E}(\Phi(\hat{\varphi})) \geq \mathcal{E}(\Phi(\hat{\rho})) \qquad \mathcal{E}_{tot}(\Phi(\hat{\varphi})) \geq \mathcal{E}_{tot}(\Phi(\hat{\rho})) \qquad \mathcal{F}^\beta(\Phi(\hat{\varphi})) \geq \mathcal{F}^\beta(\Phi(\hat{\rho}))$$

Sketch of the proof

We use the following result: for any multimode PI-BGC Φ the output coherent states majorize all other states $\Phi(\hat{\varphi}) \succ \Phi(\hat{\rho})$, for the one mode case the result holds for thermal displaced states at fixed input entropy.

So we know that output coherent state are the most *ordered* possible output states, we now just need to prove that we can always choose a coherent state that maximizes the output energy, but for any state (even non Gaussian) we have that $\mathrm{Tr}[\hat{\rho}\hat{H}] = |m|^2/2 + 1/4\mathrm{Tr}[\sigma] - n/2$, so from this expression by using peculiar properties of PI-BGCs we can prove that there exist a coherent state ϕ that maximizes the output energy.

Results on one-mode BGCs

Due to the explicit presence of squeezing operation the preoblem becomes much more intricate, in fact we cannot find a state that optimizes both terms in our functionals.

We find the best input state with fixed energy for input Gaussian state in the one-mode case.

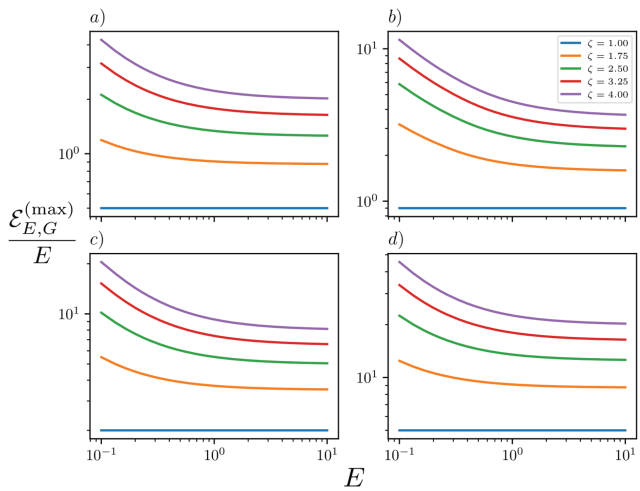


Fig. 1: Rescaled maximum output ergotropy values $\mathcal{E}^{(\max)}_{E,G}/E$ attainable with Gaussian inputs with input energy E for one-mode, not-PI BGCs. *a)* and *b)* attenuator-squeezing channels $\Gamma_{\eta,\zeta} = \mathcal{L}_{\eta,0} \circ \Sigma_\zeta$ with $\eta = 0.5$ and $\eta = 0.9$ respectively; *c)* and *d)* amplifier-sqeezing channels $\Theta_{\mu,\zeta} = \mathcal{A}_{\mu,0} \circ \Sigma_\zeta$ with $\mu = 2$ and $\mu = 5$ respectively. In the no-squeezing $\zeta = 1$ regime (blue lines) the maps are PI and the reported values coincide with the absolute maxima (η and μ) dictated by Theorem **1**.