In this preliminary exercise sheet, we will recap on some concepts that you might have learned in previous courses.

Exercise 1. Unitary and Hermitian operators

Let $\mathcal{H}$ be a Hilbert space and $A, B \in End(\mathcal{H}, \mathcal{H})$ operators in that Hilbert space. Here you have to prove some of their properties.
Note: the operator exponential is given by the power series:

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

(a) Show that $(e^A)^\dagger = e^{A^\dagger}$.

(b) Suppose that $[A, B] = AB - BA = 0$, that is, the operators A and B commute. Prove that $e^{A+B} = e^A e^B$.

(c) Show that if the operator A is Hermitian ($A = A^\dagger$) , then $U = e^{iA}$ is unitary ($UU^\dagger = U^\dagger U = \mathbb{I}$). Show also that for a collection $\{A_j\}_j$ of Hermitian operators , $U = \bigotimes_j e^{iA_j}$ is unitary.

Hinweis: Make use of the results in (a) and (b).

(d) Show that if U is a unitary, then there exists a Hermitian operator A such that $U = e^{iA}$.

(e) Suppose that V is both unitary and Hermitian. Show that the only possible eigenvalues for V are ± 1 and that $V^2 = \mathbb{I}$.

(f) Show that adding $\alpha \mathbb{I}$, where $\alpha \in \mathbb{R}$, to a Hamiltonian of a system only induces a global phase, and thus we can always shift the energy of the ground state of the Hamiltonian to zero.

(g) Suppose that A and B are Hermitian operators which commute $[A, B] = 0$. Show that in that case there exists a basis in which both A and B are diagonal, or block-diagonal.

Exercise 2. Trace and partial trace

The trace of an operator $A : \mathcal{H} \to \mathcal{H}$ is defined as $\mathrm{Tr}\,(A) = \sum_j \langle j|A|j\rangle$, where $\{|j\rangle\}_j$ is an orthonormal basis in $\mathcal{H}$. Show that the trace operation is:

(a) Linear: $\mathrm{Tr}\,(\alpha A + \beta B) = \alpha\,\mathrm{Tr}\,(A) + \beta\,\mathrm{Tr}\,(B)$ for all operators A, B and coefficients $\alpha, \beta \in \mathbb{C}$;

(b) Cyclic: $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(BCA)$ for all operators A, B, C;

(c) Basis-independent: $\mathrm{Tr}\left(UAU^\dagger\right) = \mathrm{Tr}\,(A)$ for all operators A and arbitrary unitaries U.

The partial trace is an important concept in the quantum mechanical treatment of multi-partite systems, and it is the natural generalisation of the concept of marginal distributions in classical probability theory. Let ρ_{AB} be a density matrix on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We define the *reduced state* (or *marginal*) on $\mathcal{H}_A$ as the partial trace over $\mathcal{H}_B$,

$$\rho_A := \mathrm{Tr}_B(\rho_{AB}) = \sum_j (\mathbb{I}_A \otimes \langle j|_B)\ \rho_{AB}\ (\mathbb{I}_A \otimes |j\rangle_B),$$

where $\{|j\rangle_B\}_j$ is an orthonormal basis of $\mathcal{H}_B$.

(d) Show that ρ_A is a valid density operator by proving it is:

  (i) Hermitian: $\rho_A = \rho_A^\dagger$.
  (ii) Positive: $\rho_A \geq 0$.
  (iii) Normalised: $\mathrm{Tr}\,(\rho_A) = 1$.

(e) Calculate the reduced density matrix of system A in the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \quad \text{where} \quad |ab\rangle = |a\rangle_\mathrm{A} \otimes |b\rangle_\mathrm{B}. \tag{1}$$

(f) Consider a classical probability distribution P_{XY} with marginals P_X and P_Y.

  (i) Calculate the marginal distribution P_X for

$$P_{XY}(x,y) = \begin{cases} 0.5 & \text{for } (x,y) = (0,0), \\ 0.5 & \text{for } (x,y) = (1,1), \\ 0 & \text{else}, \end{cases} \tag{2}$$

  with alphabets $\mathcal{X}, \mathcal{Y} = \{0,1\}$.
  (ii) How can we represent P_{XY} in form of a quantum state?
  (iii) Calculate the partial trace of P_{XY} in its quantum representation.

(g) Can you think of an experiment to distinguish the bipartite states of parts (b) and (c)?

Exercise 3. Composability of thermal states

Given a system with Hamiltonian

$$H = \sum_i E_i |i\rangle\langle i|,$$

and a temperature T, we define the *thermal state*

$$\tau(T) = \frac{e^{-\frac{H}{kT}}}{Z},$$

where k is a constant (Boltzmann constant), and Z is the normalization factor which is called the *partition function*:

$$Z(T,H) = \sum_i e^{-\frac{E_i}{kT}}.$$

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two systems with the joint Hamiltonian

$$H_{AB} = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B \quad \text{(the systems don't interact)}$$

(a) Show that in this case the thermal state of the joint system can be written as a tensor product of thermal states on individual subsystems:

$$\tau_{AB} = \tau_A \otimes \tau_B, \quad \text{or} \quad \frac{e^{-\frac{H_{AB}}{kT}}}{Z_{AB}} = \frac{e^{-\frac{H_A}{kT}}}{Z_A} \otimes \frac{e^{-\frac{H_B}{kT}}}{Z_B}$$

(b) Generalize the statement in (a) for the thermal state of n non-interacting subsystems.

Exercise 4. Energy preservation

Suppose that the system is characterized by a Hamiltonian H, and a unitary operation U is applied.

(a) Show that if $[U, H] = 0$, then the unitary preserves the energy of the system.

(b) Consider a four-level system with a Hamiltonian $H = \Delta|1\rangle\langle 1| + \Delta|2\rangle\langle 2| + 2\Delta|3\rangle\langle 3|$, written in the energy eigenbasis $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. Come up with one non-trivial unitary U_{pres} which would preserve the energy of the system for any state, and identify the common eigenbasis of U and H. Find another unitary $U_{\text{non-pres}}$ which would not preserve the energy of the system for any state.

(c) Give an example of an initial state of the system, for which the energy would still be preserved after applying $U_{\text{non-pres}}$.