Quantum Thermodynamics summer school
Solution 2 — Monday

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Les Diablerets

In this exercise sheet, we will explore the information-processing aspects of thermodynamics.

Exercise 1. *The non-equilibrium quantum free energy*

Consider a quantum system in state ρ and with Hamiltonian H. Its non-equilibrium quantum free energy is a functional defined as

$$F(\rho) = kTS(\rho\|\tau) - kT \log Z, \tag{1}$$

where $\tau = e^{-\beta H}/Z$ is the thermal state you saw in the preliminaries, with $\beta = (kT)^{-1}$ the inverse temperature and $Z = \mathrm{Tr}[e^{-\beta H}]$ is the partition function. $S(X\|Y)$ is the quantum relative entropy

$$S(X\|Y) = \mathrm{Tr}[X(\log X - \log Y)]. \tag{2}$$

In this exercise we will study some of its properties.

- *(a) For what states does the non-equilibrium free energy recover its equilibrium version?*
- *(b) Recall from equilibrium thermodynamics that the Helmholtz free energy is defined as $F_{eq} = U - TS$, with U the internal energy and S the entropy. Can you write an analogous expression for the non-equilibrium free energy, involving the (quantum) average energy and the von Neumann entropy?*
- *(c) Prove that the non-equilibrium contribution $kTS(\rho\|\tau)$ is always non-negative.*

  Hint: Given the function $\theta(x, y) = x \log(x/y)$, it holds that

  $$\theta(x_1 + x_2, y_1 + y_2) \leq \theta(x_1, y_1) + \theta(x_2, y_2). \tag{3}$$

- *(d) Prove that for a bipartite state ρ_{AB} one can neatly split its non-equilibrium free energy into a component from A only, one from B only and a term measuring the overall correlations between A and B (Hint: look up quantum mutual information).*
- *(e) Take an initial state ρ_S. Suppose you are allowed to bring in any thermal state τ_B at fixed temperature T (but with arbitrary Hamiltonian H_B) and perform an arbitrary unitary interaction U_{SB} involving both S and B. Before and after this interaction, the total Hamiltonian is $H_S \otimes I_B + I_S \otimes H_B$. Define the average extracted work W as the average overall energy change under this process. Prove that $W \leq F(\rho_S) - F(\rho'_S)$, where ρ'_S in the state of S after the interaction. Compare with the situation in equilibrium thermodynamics and intuitively discuss when the quantum bound can be saturated.*

Exercise 2. *Properties of thermal operations*

We defined in class the set of thermal operations $\mathcal{C}$ on a system with Hamiltonian H_S and with respect to a background (inverse) temperature β. Here we will prove some of its core properties.

- *(a) (Thermal fixed point) Prove that the thermal state is fixed: $\mathcal{C}(\tau_S) = \tau_S$, where $\tau_S = e^{-\beta H_S}/Z_S$.*
- *(b) (Time-translation symmetry) Prove that the following symmetry property is satisfied for every ρ_S and time t:*

  $$\mathcal{C} \circ \mathcal{U}_t(\rho_S) = \mathcal{U}_t \circ \mathcal{C}(\rho_S), \tag{4}$$

  where $\mathcal{U}_t(X) = e^{-iH_S t} X e^{iH_S t}$.
- *(c) (Coherence non-generation) Prove that for every state incoherent state ρ_{SA} with Hamiltonian $H_S \otimes I_A + I_S \otimes H_A$, one has that $\mathcal{C} \otimes \mathcal{I}_A(\rho_{SA})$ is incoherent too ($\mathcal{I}_A$ is the identity channel).*

- *(d)* *(Mode maps into mode) For every mode of coherence ω, and every state ρ_S*

$$\mathcal{C}(\rho_S^{(\omega)}) = \mathcal{C}(\rho_S)^{(\omega)}. \tag{5}$$

- *(e)* *Are properties (b), (c), (d) equivalent?*
- *(f)* *(Convexity) Prove that thermal operations are a convex set, meaning that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are thermal operations also $p\mathcal{C}_1 + (1-p)\mathcal{C}_2$ for $p \in [0,1]$ is a thermal operation.*

Solution

EXERCISE 1

(a) For thermal state $\rho = \tau$, $S(\tau|\tau)=0$

so $F(\tau) = -kT\log Z$

(b) $F(\rho) = kT\left(\mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\rho\log\tau]\right)$

$- kT\log Z$

$S(\rho) = -\mathrm{Tr}[\rho\log\rho]$

$$= -kT\,S(\rho) - kT\,\mathrm{Tr}\left[\rho\log\frac{e^{-H/kT}}{Z}\right] - kT\log Z$$

$$= \mathrm{Tr}[\rho H] - kT\,S(\rho) = \langle H\rangle_\rho - kT\,S(\rho)$$

(c) $\rho = \sum_i p_i |x_i\rangle\langle x_i|$ $\quad\sigma = \sum_j q_j |y_j\rangle\langle y_j|$ spectral decomposition

$$S(\rho\|\sigma) = \mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\rho\log\sigma]$$

$$= \sum_i p_i\log p_i - \sum_{i,j} p_i\log(q_j)\,|\langle x_i|y_j\rangle|^2$$

convex log

$$\geq \sum_i\Big[p_i\log p_i - p_i\log\underbrace{\sum_j q_j\,|\langle x_i|y_j\rangle|^2}_{r_j}\Big]$$

$$= \sum_i p_i\log\frac{p_i}{r_j} \; != S(\underline{p}\|\underline{r})$$

$\Theta(\alpha,\beta) = \alpha \log \frac{\alpha}{\beta}$ $\alpha > 0,\ \beta > 0$

$\Theta(\alpha_1 + \alpha_2 , \beta_1 + \beta_2) \leq \Theta(\alpha_1, \beta_1) + \Theta(\alpha_2, \beta_2)$

Then

$S(\underline{p} \| \underline{r}) = \sum_i \Theta(p_i, r_i)$

$\geq \Theta\left(\sum_i p_i , \sum_i r_i\right) = \Theta(1, 1) = 0.$

So $S(\rho \| \sigma) \geq 0.$

(d) $H_{AB} = H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B$

so $\tau_{AB} = \tau_A \otimes \tau_B$

$F(\rho_{AB}) = kT S(\rho_{AB} \| \tau_A \otimes \tau_B) - kT \log Z_{AB}$

$= -kTS(\rho_{AB}) - kT \operatorname{Tr}[\rho_{AB} \log \tau_A \otimes \tau_B]$

$- kT \log Z_A - kT \log Z_B$

$= kT(S(\rho_A) + S(\rho_B) - S(\rho_{AB}))$

$- kT \operatorname{Tr}[\rho_A \log \tau_A] - kT \log Z_A - kTS(\rho_A)$

$$- kT\,\mathrm{Tr}\left[\rho_B \log \tau_B\right] - kT \log Z_B - kT S(\rho_B)$$

$$= kT \underset{\text{mutual info}}{I(A:B)} + F(\rho_A) + F(\rho_B)$$

(e) $\tau_{SB} = \tau_S \otimes \tau_B$ since $H_{SB} = H_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes H_B$

$$W_{EXT} := \mathrm{Tr}\left[H_{SB}\left(\rho_S \otimes \tau_B - U \rho_S \otimes \tau_B U^\dagger\right)\right]$$

unitary isoentropic

$$= \mathrm{Tr}\left[H_{SB}\left(\rho_S \otimes \tau_B\right)\right] - kT S(\rho_S \otimes \tau_B)$$

$$- \mathrm{Tr}\left[H_{SB} U \rho_S \otimes \tau_B U^\dagger\right] + kT S\left(U \rho_S \otimes \tau_B U^\dagger\right)$$

$$= F(\rho_S \otimes \tau_B) - F\left(U(\rho_S \otimes \tau_B) U^\dagger\right).$$

Let $\rho'_S = \mathrm{Tr}_B\, \rho'_{SB}$ $\quad \rho'_{SB} = U(\rho_S \otimes \rho_B) U^\dagger$

$\rho'_B = \mathrm{Tr}_S\, \rho'_{SB}$

From (d)

mutual info ρ'_{SB} ↓

$$F\left(U(\rho_S \otimes \tau_B) U^\dagger\right) = F(\rho'_S) + F(\tau'_B) - kT I(S':B')$$

$$\leq F(\rho'_S) + F(\tau_B)$$

↑ mutual info non-negative
F(x) minimal for thermal state

We conclude from $F(\rho_S \otimes \tau_B) = F(\rho_S) + F(\tau_B)$

$$W_{EXT} \leq F(\rho_S) + \cancel{F(\tau_B)} - F(\rho_S') - \cancel{F(\tau_B)}.$$

EXERCISE 2

(a) $\mathcal{E}(\tau_S) =$

$$= \mathrm{Tr}_B\left[U(\tau_S \otimes \tau_B)U^\dagger\right]$$

But $[U, H_S + H_B] = 0$

$$\Rightarrow [U, \tau_S \otimes \tau_B] = 0$$

so

$$\mathcal{E}(\tau_S) = \mathrm{Tr}_B\left[(\tau_S \otimes \tau_B)\overbrace{UU^\dagger}^{\mathbb{1}}\right] = \tau_S.$$

(b) $\mathcal{U}_t^S(X) := e^{-iH_S t} X e^{iH_S t}$

$\mathcal{U}_t^B(X) := e^{-iH_B t} X e^{iH_S t}$

$$\mathcal{E}(\mathcal{U}_t^S(\rho_S)) = \mathrm{Tr}_B\left[U(\mathcal{U}_t^S(\rho_S) \otimes \tau_B)U^\dagger\right] =$$

$$\underset{\nearrow}{=} \mathrm{Tr}_B\left[U(\mathcal{U}_t^S(\rho_S) \otimes \mathcal{U}_t^B(\tau_B))U^\dagger\right]$$

thermal
state stationary

$$= \mathrm{Tr}_B\left[U\, \mathcal{U}^t_S \otimes \mathcal{U}^t_B (\rho_S \otimes \tau_B)\, U^\dagger\right]$$

$$= \mathrm{Tr}_B\Big[\underbrace{U\, e^{-i(H_S+H_B)t}}_{\text{commute}} \rho_S \otimes \tau_B\, \underbrace{e^{i(H_S+H_B)t}\, U^\dagger}_{\text{commute}}\Big]$$

$$\underset{\uparrow}{=} e^{-iH_S t}\, \mathrm{Tr}_B\left[U(\rho_S \otimes \tau_B)U^\dagger\right] e^{iH_S t}$$

cyclic property
of trace on B

$$= \mathcal{U}^S_t(\mathcal{E}(\rho_S)).$$

(c) CLAIM

&

(e)

$\mathcal{E}_S \otimes \mathcal{I}_A(\rho_{SA})$ incoherent for all $H_{SA} = H_S + H_A$ and all ρ_{SA}

$$\Leftrightarrow \quad \mathcal{E}_S \circ \mathcal{U}_t = \mathcal{U}_t \circ \mathcal{E}_S$$

Proof ($\Rightarrow$) In particular,

$J_{SS'} = (\mathcal{E}_S \otimes \mathcal{I}_{S'})(\Phi^+_{SS'})$ is incoherent with

$$H_{SA} = H \otimes \mathbb{1} - \mathbb{1} \otimes H^* \quad (S \simeq S')$$

By the inverse Choi map

$$\mathcal{E}(\rho_{S'}) = \mathrm{Tr}_{S'}\left[J_{SS'}\left(\mathbb{1} \otimes \rho^T_{S'}\right)\right]$$

Also, $\mathcal{U}_S \otimes \mathcal{U}^T_{S'}(J_{SS'}) \underset{\tau}{=} e^{i(H\otimes 1\!\!1 - 1\!\!1\otimes H^*)t} J_{SS'}\, e^{-i(H\otimes 1\!\!1 - 1\!\!1 \otimes H^*)t} = J_{SS'}$

$\tau = (t)^*$

Then

$$\mathcal{U}_S \circ \mathcal{E}(\rho_{S'}) = \mathrm{Tr}_{S'}\left[(\mathcal{U}_S \otimes \mathcal{J}_{S'})(J_{SS'})\,(1\!\!1_S \otimes \rho^T_{S'})\right] =$$

$$= \mathrm{Tr}_{S'}\left[(1\!\!1_S \otimes \mathcal{U}^*_{S'})\underbrace{(\mathcal{U}_S \otimes \mathcal{U}^T_{S'})(J_{SS'})}_{J_{SS'}}(1\!\!1_S \otimes \rho^T_{S'})\right]$$

$$= \mathrm{Tr}_{S'}\left[J_{SS'}\,(1\!\!1_S \otimes \mathcal{U}_{S'}(\rho_{S'})^T)\right] = \mathcal{E}\circ\mathcal{U}_{S'}(\rho_{S'})$$

$$(\Longrightarrow)\quad \mathcal{U}_S \otimes \mathcal{U}_A\,(\mathcal{E}_S \otimes \mathcal{J}_A)(\rho_{SA}) = (\mathcal{E}_S \otimes \mathcal{J}_A)(\mathcal{U}_S \otimes \mathcal{U}_A)(\rho_{SA})$$

$$= (\mathcal{E}_S \otimes \mathcal{J}_A)(\rho_{SA}).$$

This shows coherence non-gen and covariance are equivalent.

Since TO are covariant (see (b)), they satisfy coherence non-gen.

(d) We did not have time to cover this! See Def 11 & Theorem 13 of arXiv 1807.11549. This also completes the answer to question (e). Properties (b), (c), (d) are equivalent.

(f) Appendix C arXiv 1410.4572.

Exercise 3. ***Constructing thermal monotones***

Suppose that $\sigma = \mathcal{C}(\rho)$*, where* $\mathcal{C}$ *is a thermal operation. Let* $\vec{x}$ *and* $\vec{y}$ *be the vectors of populations (occupation of energy levels) of* ρ *and* σ*, respectively.*

For any h *convex function in* $\mathbb{R}$*, define the* f*-divergence*

$$f(\vec{x}) = \sum_i g_i h(x_i/g_i), \tag{6}$$

where $g_i = e^{-\beta E_i}/Z$*. Prove that*

$$f(\vec{x}) \geq f(\vec{y}), \tag{7}$$

so that each f *can be interpreted as a generalized free energy.*

Solution This is a tool to construct $\vec{g}$-Schur-convex functions on $(\mathbb{R}^n, \succ)$ from convex functions on $\mathbb{R}$: Let $h : \mathbb{R} \to \mathbb{R}$ be convex. Then the function f

$$f(\vec{x}) = \sum_{i=1}^n g_i h\left(\frac{x_i}{g_i}\right),$$

is $\vec{g}$-Schur-convex. We prove the statement for h convex. As we have seen, $\vec{x} \succ_g \vec{y}$ if and only if $y_i = \sum_{j=1}^n G_{i|j} x_j$, with $\sum_{j=1}^n G_{i|j} \frac{g_j}{g_i} = 1$ and $\sum_{i=1}^n G_{i|j} = 1$. Then,

$$f(\vec{y}) = \sum_{i=1}^n g_i h\left(\sum_{j=1}^n G_{i|j} \frac{x_j}{g_i}\right) = \sum_{i=1}^n g_i h\left(\sum_{j=1}^n \left[G_{i|j} \frac{g_j}{g_i}\right] \frac{x_j}{g_j}\right) \leq \sum_{j=1}^n g_j h\left(\frac{x_j}{g_j}\right) = f(\vec{x}).$$

Exercise 4. ***Work cost of processes on a quantum computer***

Consider a quantum computer with a register of n *qubits described by the trivial Hamiltonian* $H = 0$*. Assume that the quantum computer can perform standard unitary gates at no work cost on the register. Suppose furthermore that the quantum computer has two functions* RESET_ZERO_k *and* EXTRACT_WORK_k *defined as follows. The function* RESET_ZERO_k *transforms the* k*-th qubit to the* $|0\rangle$ *state, regardless of the* k*-th qubit's initial state, at a work cost equal to* $kT \ln 2$ *where* T *is some fixed temperature and* k *is Boltzmann's constant. The function* EXTRACT_WORK_k *is such that if the* k*-th qubit is in the* $|0\rangle$ *state, then it is transformed to the maximally mixed state while extracting* $kT \ln 2$ *work. (I.e., the functions* RESET_ZERO_k *and* EXTRACT_WORK_k *operate on the* k*-th qubit like the reset procedure and the work extraction process do on a Szilárd engine.)*

(a) *Suppose that when you turn on your quantum computer, the memory register is uninitialized. We model this situation by assuming that the register is in a maximally mixed state. How much work is required to restore the memory register to the computational all-zero state* $|0\rangle^{\otimes n}$*? How about if you would like to initialize it in the all-one state* $|1\rangle^{\otimes n}$*? How about if you would like to prepare a given fixed computational basis state* $|x_1\rangle|x_2\rangle \ldots |x_n\rangle$*?*

Solution Resetting n qubits from a maximally mixed state to the all-zero state costs $kT\ln(2)$ work per bit, according to Landauer's principle, for a total cost of $nkT\ln(2)$.

The same principle applies whether you'd like to reset to one or reset to zero, costing $kT\ln(2)$ work per qubit. So you need to pay $nkT\ln(2)$ regardless of which pure state you'd like to reset to, such as the all-one state or any fixed computational basis state $|x_1\rangle|x_2\rangle\ldots|x_n\rangle$.

(b) *Find protocols and their associated work cost in order to achieve the following tasks:*

(i) *Suppose three qubits of the memory are entangled in the GHZ state*

$$|GHZ\rangle = \frac{(|000\rangle + |111\rangle)}{\sqrt{2}}.$$

Find a protocol that resets the first qubit to $|0\rangle$ *while leaving the reduced state of the second and third qubits unchanged.*

Solution Here is a possible protocol (other protocols are possible):

1. Apply a C-NOT gate on qubits 2 and 3 to obtain the state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{1,2} \otimes |0\rangle_3 \ ; \tag{S.1}$$

2. Extract work from the first two qubits, leaving them maximally mixed (extract $2 \times kT\ln(2)$ work). The state is now

$$\frac{1}{4}\mathbb{1}_{1,2} \otimes |0\rangle\langle 0|_3 \ ; \tag{S.2}$$

3. Reset the first qubit to zero, paying $kT\ln(2)$ work. The state is now

$$|0\rangle\langle 0|_1 \otimes \frac{1}{2}\mathbb{1}_2 \otimes |0\rangle\langle 0|_3 \ ; \tag{S.3}$$

4. Apply a C-NOT gate on the second and third qubits to obtain

$$|0\rangle\langle 0|_1 \otimes \left(\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|\right)_{2,3} \ , \tag{S.4}$$

achieving the desired transformation. In total, we've extracted $kT\ln(2)$ work.

This protocol is optimal as can be seen by the entropic bound

$$\begin{aligned}\frac{W_{\text{cost}}}{kT\ln(2)} &= H_{\max,0}(1|2,3) = \log\left\|\operatorname{tr}_1\left(|GHZ\rangle\langle GHZ|\right)\right\|_\infty \\ &= \log\left\|\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|\right\|_\infty = \log\frac{1}{2} = -1 \ ,\end{aligned} \tag{S.5}$$

noting that the projector onto the support of $|GHZ\rangle\langle GHZ|$ is simply $|GHZ\rangle\langle GHZ|$.

(ii) *Suppose three qubits of the memory are in the classically correlated state*

$$\rho = \frac{1}{3}|000\rangle\langle 000| + \frac{2}{3}|111\rangle\langle 111|.$$

Find a protocol that resets the first qubit to $|0\rangle$ *while leaving the reduced state of the second and third qubits unchanged.*

Solution Here is a possible protocol (other protocols are possible):

1. Apply a C-NOT gate between the first and second qubit to obtain the state

$$|0\rangle\langle 0|_1 \otimes \Big(\frac{1}{3}|00\rangle\langle 00| + \frac{2}{3}|11\rangle\langle 11|\Big) \ , \tag{S.6}$$

achieving the desired process at no work cost.

This protocol is optimal, as can be seen by the entropic bound. The projector onto the support of the state is

$$\Pi^{\rho}_{1,2,3} = |000\rangle\langle 000| + |111\rangle\langle 111| \ . \tag{S.7}$$

The entropic bound then reads

$$\begin{aligned}\frac{W_{\text{cost}}}{kT\ln(2)} &= H_{\max,0}(1|2,3) = \log\Big\|\operatorname{tr}_1\big(|000\rangle\langle 000| + |111\rangle\langle 111|\big)\Big\| \\ &= \log\Big\||00\rangle\langle 00| + |11\rangle\langle 11|\Big\| = \log(1) = 0 \ . \end{aligned}\tag{S.8}$$

(iii) Implement a MEASURE$_{k,\ell}$ *function that will set the ℓ-th qubit to the classical value of the k-th qubit in the computational basis, i.e., such that:*

$$\text{MEASURE}_{k,\ell}\big(|i\rangle\langle i|_k |j\rangle\langle j|_\ell\big) = |i\rangle\langle i|_k |i\rangle\langle i|_\ell \quad \forall i,j.$$

Solution Here's a possible protocol:

1. Reset the qubit ℓ to the $|0\rangle$ state, paying $kT\ln(2)$ work.
2. Apply a C-NOT gate between the k-th and the ℓ-th qubits, thus correlating the ℓ-th qubit with the value of the k-th qubit. This achieves the desired process.

Note that the total work cost of this measurement process is $kT\ln(2)$. The only work that was required was to prepare the ancilla qubit that will store the measurement result, the process itself of correlating the ancilla with the k-th qubit doesn't require work.

We can write the process we want to implement as

$$\mathcal{E}((\cdot)_{k,\ell}) = \sum_{i,j} \langle i,j\,|\,(\cdot)\,|\,i,j\rangle_{k,\ell}\, |i,i\rangle\langle i,i|_{k,\ell} \ . \tag{S.9}$$

(The protocol above doens't exactly implement this process, because it handles states with coherences differently. To achieve the process $\mathcal{E}$, you can simply add a dephasing operation to make sure the output is classical; the dephasing costs no additional work.)

The minimal work cost is then given (for any full-rank σ) by

$$\begin{aligned}W_{\text{cost}}(\mathcal{E},\sigma) &= \log\big\|\mathcal{E}(\mathbb{1}_{k,\ell})\big\| = \log\Big\|\sum_{i,j}\langle i,j\,|\,\mathbb{1}\,|\,i,j\rangle_{k,\ell}\,|i,i\rangle\langle i,i|_{k,\ell}\Big\| \\ &= \log\Big\|2\sum_i |i,i\rangle\langle i,i|_{k,\ell}\Big\| = \log 2 = 1 \ , \end{aligned}\tag{S.10}$$

where the factor 2 comes from the summation over j.

(c) Using the entropic bounds introduced in the lecture, determine the optimal work cost of the tasks (i), (ii), and (iii). Are the protocols you found optimal? If not, try to find optimal ones.

Solution See solutions in part (b).

Exercise 5. *Entropy and Observers in Gibbs' paradox*

When two different gases mix, the system acquires a contribution to the entropy, the entropy of mixing, *in addition to the entropy associated with each gas. Gibbs' paradox refers to the discontinuity when we take the gases to be arbitrarily similar. At which point are they no longer two different gases and is the entropy of mixing no longer present? Here, we study an example presented in Jaynes' excellent treatment of the subject.*[1]

(a) A box contains two volumes V_1 and V_2 of identical argon gases at the same pressure and temperature. We remove the separator that keeps the gases apart, allowing them to mix freely. What is the entropy change of the full system?

Solution Thermodynamics ascribes zero entropy change to the mixing of two samples of argon at the same temperature and pressure.

(b) Suppose that the two volumes of argon gas are in fact two different kinds of argon, $A1$ and $A2$. But these two different types of argon haven't been discovered yet and it still looks like we're mixing two volumes of identical argon gases. $A1$ and $A2$ are identical in all physical aspects (mass, charge, etc.), except that $A2$ is soluble in Whifnium, a hypothetical substance that hasn't been discovered yet. Consider the mixing process as in (a). What is the entropy change of the full system? Explain your answer.

Solution Even if we manage to get pure $A1$ in volume V_1, and pure $A2$ in volume V_2, we would have no way of detecting any difference in the resulting diffusion process, so we would still ascribe zero entropy change to the process.

(c) 100 years later, we've discovered Whifnium and we know there are two types of Argon, $A1$ and $A2$. Consider the mixing process as in (a). What is the entropy change of the full system? Explain your answer.

Solution Now we can prepare two volumes of $A1$ and $A2$. We start the experiment with $n_1 = \alpha n$ moles of $A1$, and $n_2 = (1-\alpha)n$ moles of $A2$, and due to our greater knowledge the diffusion results in entropy increase

$$\Delta S = \Delta S_1 + \Delta S_2 \text{ with}$$
$$\Delta S_1 = -n\alpha \log \alpha,\ \Delta S_2 = -n(1-\alpha)\log(1-\alpha).$$

If this entropy is more than a consequence of our knowledge increase, it has to have observable consequences, like work we could extract from the process. All in all, then the amount of useful work we can extract from a system depends on our "subjective" information about the state.

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[1] `https://link.springer.com/chapter/10.1007/978-94-017-2219-3_1`