Exercise 1. A trio: unbiased, faithful and non-invasive

Recall the three fundamental properties of an ideal measurement, effecting the transformation $\rho_S \otimes \rho_P \to \tilde{\rho}_{SP}$, which were presented as definitions in the lecture

(i) **Unbiased**. The probability of finding the pointer in a state associated with outcome i after the interaction is the same as the probability of finding the system in the state $|i\rangle_S$ before the interaction,

$$\mathrm{tr}\big[\mathbb{I} \otimes \Pi_i \tilde{\rho}_{SP}\big] = \mathrm{tr}\big[\,|i\rangle\langle i|_S \rho_S\big] = \rho_{ii} \quad \forall i \; \forall \rho_S. \tag{1}$$

A measurement is unbiased if the pointer reproduces the measurement statistics of the system.

(ii) **Faithful**. There is a one-to-one correspondence between the pointer outcome and the post-measurement system state

$$C(\tilde{\rho}_{SP}) := \sum_i \mathrm{tr}\big[\,|i\rangle\langle i| \otimes \Pi_i \;\tilde{\rho}_{SP}\big] = 1 \quad \forall \rho_S, \tag{2}$$

i.e., $\tilde{\rho}_{SP}$ has perfect correlation: on observing the pointer outcome i (associated to Π_i), one concludes that the system is left in the state $|i\rangle_S$ with certainty.

(iii) **Non-invasive**. The probability of finding the system in the state $|i\rangle_S$ is the same before and after the interaction with the pointer,

$$\mathrm{tr}\big[\,|i\rangle\langle i|_S \tilde{\rho}_{SP}\big] = \mathrm{tr}\big[\,|i\rangle\langle i|_S \rho_S\big] = \rho_{ii} \;\; \forall\, i \; \forall \rho_S. \tag{3}$$

This property only holds for the basis $|i\rangle_S$ and coherences appearing on the off-diagonal can, in general, be destroyed.

(a) Consider the initial single-qubit system state $\rho_S = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$, and the two-qubit final state $\tilde{\rho}_{SP} = \sum_{m,n=0,1} |m,n\rangle\langle m,n|$ with $p_{01} = p_{10} = p_{11} = \frac{1}{8}$ and $p_{00} = \frac{5}{8}$. For $\Pi_i = |i\rangle\langle i|_P$, which of the properties (i)–(iii) hold?

Consider a measurement using a single pointer qubit and assume that by some means it has been prepared in the ground state $\rho_P = |0\rangle\langle 0|_P$. We model the interaction with the pointer by applying a controlled NOT operation $U_{\text{CNOT}} = |0\rangle\langle 0|_S \otimes \mathbb{1}_P + |1\rangle\langle 1|_S \otimes X_P$, with the usual Pauli operator $X = |0\rangle\langle 1| + |1\rangle\langle 0|$. Denoting the matrix elements of the initial state as $\rho_{ij} = \langle i|\rho_S|j\rangle$, we can then write the post-measurement state $\tilde{\rho}_{SP}$ as

$$\tilde{\rho}_{SP} = U_{\text{CNOT}}\, \rho_{SP} U^\dagger_{\text{CNOT}} = \sum_{i,j=0,1} \rho_{ij} |ii\rangle\langle jj|\,. \tag{4}$$

The system and pointer are now perfectly (classically) correlated: whenever the pointer is found in the state $|0\rangle_P$ ($|1\rangle_P$), the system is left in the corresponding state $|0\rangle_S$ ($|1\rangle_S$). For the choice $\Pi_i = |i\rangle\langle i|_P$, we find that the measurement is faithful, i.e., that it satisfies Property (ii).

In the lecture, we argued that pure state pointers are not physically feasible. We now assume that the pointer has been prepared at some finite non-vanishing temperature $T = 1/\beta$, such that $\rho_P = p|0\rangle\langle 0|_P + (1-p)|1\rangle\langle 1|_P$ for some $p = (1 + e^{-\omega\beta})^{-1} = \mathcal{Z}^{-1}$ with $0 < p < 1$.

(b) Compute the correlation function $C(\tilde{\rho}_{SP})$.

(c) Is U_{CNOT} biased or unbiased in general?

Now replace U_{CNOT} by $U_{\text{unb.}} = |00\rangle\langle 00| + |01\rangle\langle 11| + |11\rangle\langle 10| + |10\rangle\langle 01|$

(d) Recompute $C(\tilde{\rho}_{SP})$ and comment on which of the Properties (i)–(iii) are satisfied.

Fix ρ_S to be some particular state and consider a measurement interaction between it and a pointer. Show that the following relations for the measurement hold:

(e) faithful AND unbiased $\rightarrow$ non-invasive.

(f) faithful AND non-invasive $\rightarrow$ unbiased.

Now relax ρ_S to all states ($\forall \rho_S$). Show that a measurement that is unbiased and non-invasive for any possible ρ_S is also faithful, i.e.,

(g) unbiased AND non-invasive $\rightarrow$ faithful.

Exercise 2. Geroch process: perpetuum mobile out of black holes

The exercise is borrowed from ETH Theory of Heat course, and was originally designed by Giulia Mazzola.

In 1971, Robert Geroch proposed a thought experiment in a colloquium at Princeton University, according to which it would be possible to fully convert the heat of thermal radiation into mechanical work with the help of a black hole. This would contradict the second law of thermodynamics, and allow for constructing a *perpetuum mobile* of the second kind.

In order to save the thermodynamics, in this exercise we justify that black holes must have a finite temperature based on Geroch process, and calculate it explicitly.

The concept we are going to use is known as *gravitational redshift*, an effect derived from general relativity. Imagine that a physicist Alice is located near a massive object, for example, the Sun, or a black hole. She emits a beam of light of frequency ω; if this beam is received by a second physicist Bob, who is asymptotically far away from the object (where the spacetime can be considered flat), he will measure a lower frequency $\omega' < \omega$. The light beam experiences a redshift. This happens because according to GR, massive objects bend space-time, and therefore time does not pass equally in different places. The strength of the shift depends on the mass of the object, which determines the spacetime curvature.

In the following, we will consider a static, non-charged black hole. Geroch's thought experiment can be divided into four steps:

1. To start, Bob takes a box filled with thermal radiation at temperature T, so that the full box has (rest) energy m [1].

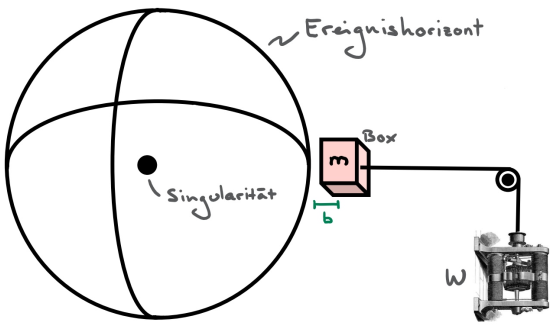
[1] In the following we use the convention $c = 1$. We also assume that the empty box is massless.

2. Bob ties the box to a mechanical system (for example, a string) and lets the box (quasi-statically) closer and closer to the black hole. If the box is at a distance r to the black hole, then the extracted work W corresponds exactly to the difference between the rest energy m and the energy that Bob would measure due to the gravitational redshift. Interestingly, it turns out that the redshift at the event horizon ($r_s = 2GM$) is infinitely large, and therefore the energy of the box disappears completely from Bob's point of view. This means that once the box reaches the event horizon, Bob has extracted the work $W = m$.

3. At this point the box is opened and the radiation is let into the black hole. Since the box has no energy, the mass M and the black hole as a whole remain unchanged.

4. After that, Bob pulls the box back. The box no longer has any energy, so he doesn't need to spend any work.

5. Finally, Bob again fills the box with thermal radiation (or lets the box thermalize at temperature T) and thus can repeat the process again, which he can use as a perpetual motion machine.



One of the possible resolutions to the paradox was suggested by Jacob Bekenstein two year later. He pointed out that the box must have a finite expansion b in order to contain thermal radiation at a finite temperature T. Therefore it is not possible to reach the event horizon exactly, but the box can only be brought to the horizon up to a distance b. At that point, the box would still have finite energy

$$E = 2mb\Theta \quad \text{with} \quad \Theta = \frac{1}{4r_s} = \frac{1}{8GM}. \tag{5}$$

(a) Consider a box with thermal radiation of temperature T. Find a reasonable estimate for the minimum size (radius) b of the box as a function of T.

(b) Calculate the efficiency of the Geroch process as a function of b. Equate it to the Carnot efficiency of a heat engine with cold and hot reservoirs at temperatures $T_{\text{hot}} = T$ and $T_{\text{cold}} = T_{\text{bh}}$. For this we want to assume that the Geroch process (modified by Bekenstein) is (almost) reversible.

(c) Use the estimate from (a) to find an expression for T_{bh}. Congrats! You have derived the black hole temperature.