Quantum Thermodynamics summer school
Exercise Sheet 4 — Thursday

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Les Diablerets

In this exercise sheet, we will explore correlations and partition functions in thermodynamic systems, and look at an example of a Lindblad master equation.

Exercise 1. An inequality for traces

In this exercise, we prove an inequality used in the lecture to prove the bound on ratios of partition functions.

Consider two Hermitian positive operators A and B.

(a) First show that $\operatorname{Tr}(AB) \leq ||B|| \operatorname{Tr}(A)$, where $||.||$ is the operator norm.
Hint: Consider $\operatorname{Tr}(AB) = \operatorname{Tr}\left(B^{1/2}AB^{1/2}\right)$.

(b) Show that

$$|\log \operatorname{Tr}\left(e^{A+B}\right) - \log \operatorname{Tr}\left(e^{A}\right)| = \left|\int_0^1 \frac{d}{dt} \log \operatorname{Tr}\left(e^{A+tB}\right) dt\right|.$$

(c) Using the results of (a) and (b), prove the following inequality

$$|\log \operatorname{Tr}\left(e^{A+B}\right) - \log \operatorname{Tr}\left(e^{A}\right)| \leq ||B||.$$

Then, we obtain the result about partition functions by choosing e.g. $A = \beta H$ and $B = \beta V$.

Exercise 2. Mutual information and correlations

The quantum mutual information is defined for a bipartite state ρ^{AB} as

$$I(A:B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}).$$

(a) Show that $I(A:B) = D(\rho^{AB}|\rho^A \otimes \rho^B)$, where $D(\rho|\sigma) = \operatorname{Tr}(\rho(\log(\rho - \sigma))$ is the quantum relative entropy.

(b) Show that the mutual information upper bounds all correlation functions

$$I(A:B) \geq \max_{||O_A||,||O_B|| \leq 1} |\operatorname{Tr}\left((O_A \otimes O_B)\rho^{AB}\right) - \operatorname{Tr}\left(O_A \rho^A\right) \operatorname{Tr}\left(O_B \rho^b\right)|. \tag{1}$$

Hint: Use Pinsker's inequality $D(\rho|\sigma) \leq \frac{||\rho-\sigma||_1^2}{2}$ *and the definition of the 1-norm.*

Exercise 3. Bounding the average fluctuations around the late-time value

In the lecture, we have seen how we can bound the correlations for the system evolving under a Hamiltonian with non-degenerate energy gaps. The same proof technique can be applied in a slightly different setting.

Suppose that we consider two observables A and B. For them, let us introduce a correlation function

$$C^{AB}(t) \equiv \langle A(t)B \rangle_\beta = \mathrm{Tr}\left(\rho A(t) B\right),$$

where the evolution is generated by a time-independent Hamiltonian H, and $A(t) = e^{iHt} A e^{-iHt}$ is the evolved observable in the Heisenberg picture. We additionally assume that the energy gaps of H are non-degenerate, and $[\rho, H] = 0$.

In the limit $t \to \infty$, we can define $C_\infty^{AB} = \lim_{T\to\infty} \int_0^T \frac{\mathrm{d}t}{T} C^{AB}(t)$, and the average fluctuations around the late-time value as

$$\sigma_C^2 = \lim_{T\to\infty} \int_0^T \frac{\mathrm{d}t}{T} \left(C^{AB}(t) - C_\infty^{AB}\right)^2.$$

(a) Expand σ_C^2 in energy eigenbasis, and show that

$$\sigma_C^2 = \sum_{j\neq k} \rho_{jj}\rho_{kk} A_{jk} A_{kj} B_{jk} B_{kj},$$

where A_{kj}, B_{jk} are matrix elements in the energy eigenbasis, $A = \sum_{jk} A_{jk} |j\rangle\langle k|$. Use the fact that the energy gaps are non-degenerate!

(b) Show that the expression above can be upper bounded by

$$\sum_{j\neq k} \rho_{jj}\rho_{kk} A_{jk} A_{kj} B_{jk} B_{kj} \leq \max_{j\neq k}\{|A_{kj}B_{jk}|\} \sum_{j\neq k} \rho_{jj}\rho_{kk} \left|A_{jk}B_{kj}\right|.$$

Notice that we can also write $\max_{j\neq k}\{|A_{kj}B_{jk}|\} \leq ||A|| ||B||$, but we would be losing an exponentially small factor in the process).

(c) Now use the Cauchy-Schwarz inequality, and prove

$$\sigma_C^2 \leq \max_{j\neq k}\{|A_{kj}B_{jk}|\} \sqrt{\mathrm{Tr}\left(\rho A \rho A\right)} \sqrt{\mathrm{Tr}\left(\rho B \rho B\right)}.$$

(d) Apply the Cauchy-Schwarz inequality yet again, and obtain

$$\sigma_C^2 \leq \|A\| \, \|B\| \max_{j\neq k}\{|A_{kj}B_{jk}|\} \, \mathrm{Tr}\left(\rho^2\right).$$

Hint: For positive operators $\mathrm{Tr}\left(PQ\right) \leq \|P\| \, \mathrm{Tr}\left(Q\right)$. *You can show this in the second exercise.*

We have successfully derived an upper bound on the late-time fluctuations

$$\sigma_C^2 \leq \|A\| \, \|B\| \max_{j\neq k}\{|A_{kj}B_{jk}|\} \, \mathrm{Tr}\left(\rho^2\right).$$

Note that the bound depends on $\mathrm{Tr}\left(\rho^2\right)$. This quantity is the inverse of what is sometimes called the *effective dimension* of the system: $d_{eff}^{-1} = \mathrm{Tr}\left(\rho^2\right)$.

(e) How does the upper bound look like for a microcanonical ensemble $\rho = \frac{\mathbb{I}}{d}$? And for a thermal state $\rho \equiv e^{-\beta H}/Z_\beta$ at inverse temperature β with partition function Z_β?

To establish a connection between free energy and this effective dimension, let us consider Rényi entropy of order q of a probability distribution p

$$S_q(p) = \frac{1}{1-q} \log \sum_i p_i^q, \quad q \in (0, \infty).$$

We can imagine the probability distribution p as describing a Gibbs state at temperature T_0 for some energies E_i such that $\sum_i \exp(-\frac{E_i}{T_0}) = 1$.

(f) The free energy is given by $F = -T \log Z$. Show that

$$S_{T_0/T} = -\frac{F}{T - T_0}.$$

(g) The quantum generalization of Rényi entropy is defined by $S_q(\rho) = \frac{1}{1-q} \log \mathrm{Tr}\,(\rho^q)$. Using previous considerations, derive a bound on $\mathrm{Tr}\,(\rho^2)$ in terms of free energy F.

Exercise 4. Weakly damped harmonic oscillator

A weakly damped harmonic oscillator with the Hamiltonian $H = \hbar\omega \hat{a}^+ \hat{a}$ (for example, coupled to a heat bath) can be described by the following Lindblad master equation for the density matrix:

$$\frac{d}{dt}\rho(t) = \frac{i}{\hbar}[\rho(t), H] + \gamma(a\rho a^+ - \frac{1}{2}a^+ a\rho - \frac{1}{2}\rho a^+ a)$$

Let's assume that the only non-zero matrix elements are ρ_{ij} with $i, j \in \{0, 1\}$. What do the equations for these matrix elements look like? What is the decay rate for the off-diagonal element ρ_{01} ("dephasing rate") ? In the general case of an arbitrary ρ , what does the equation for the matrix element ρ_{nn} look like?